decades but rather wished to demonstrate that Gauss and Laplace might have had more to offer in this instance than do modern schools. Such general and unwarranted statements as "In reality, numerous observations on many streams have shown that the distribution of the flood discharges is skew" have unfortunately destroyed the general faith in this application of the classic theory.

In connection with the computation of plotting positions, Professor Gumbel's statement that the system advocated by the writer is impracticable has no bearing on the validity of the system. The plotting positions given in Table 1 are derived by the accepted laws of probability from the only plausible basic assumption. Therefore, the system is not an invention of the writer but is rather a solution of a mathematical equation. The fact that the solution becomes laborious has no more bearing on its validity than does the fact that π is not an integer prove that it is incorrect. In recognizing the impracticability of computing plotting positions accurately, the writer has suggested an approximate method which is entirely satisfactory for all purposes. This approximate method is not a compromise between the recurrence-interval and exceedence-interval methods, as stated by Professor Gumbel. His further statement that the corrections in plotting positions now used should depend upon the distribution curve is in error since the selection of a finite number of occurrences from an infinite number, being arbitrary, is not influenced by magnitudes of the occurrences and consequently cannot be influenced by the distribution function.

The writer has been unable to make a complete check of the mathematics involved in Professor Gumbel's method and is therefore unable to evaluate its merits completely. However, from the curves published by Professor Gumbel, the method appears to be justified from an engineering standpoint. The procedure outlined by the writer, as shown in Fig. 11, also is justified from an engineering standpoint and has the additional advantages of simplicity and agreement with accepted theory.

The preceding discussions of this paper have been of great assistance to the writer in clarifying some of the important points of the paper. The contributions of those who have presented discussions are sincerely appreciated.

The writer purposely has avoided a discussion of the merits and demerits of the various "methods" of statistical analysis now employed in hydrology, as such discussions have been published many times. Rather it was intended to explain the basic logic of the duration-curve type of analysis and to call attention to a few essential respects in which the mathematical theory of the duration curve has been departed from repeatedly.

It is believed that, in recent years, the theory of the duration curve has been developed so sufficiently that its unqualified, but proper application, can be justified from an engineering standpoint. Many alternative methods used in hydrologic design, such as the application of enveloping curves and transposed storms, do not have mathematical significance. Such arbitrary designs are not justified from an engineering standpoint if the desired magnitude of a flood can be stipulated in such terms that the flood can be derived mathematically and can be given significance thereby.

1.2 NMN - Numerical

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NUMERICAL PROCEDURE FOR COMPUTING DEFLECTIONS, MOMENTS, AND BUCKLING LOADS

By N. M. Newmark,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. Bruce Johnston, M. S. Ketchum, Jr., John B. Wilbur, Ralph W. Stewart, Stefan J. Fraenkel, Alfred S. Niles, Camillo Weiss, A. A. Eremin, Myron L. Gossard, Robert A. Williamson, I. Oesterblom, C. W. Dunham, and N. M. Newmark.

Synopsis

A numerical procedure for computing the deflections and moments in beams and columns is described herein. The method is of particular applicability in determining critical buckling loads and configurations of bars of variable cross section loaded in various ways. For such problems the procedure becomes one of successive approximations. By means of a simple modification of the data entailing very little increase in numerical work, considerably greater accuracy is obtainable by this procedure than by others of similar nature hitherto available.

The numerical procedure is approximate, but leads to exact moments (or deflections) when the loading diagram (or diagram of "angle changes") is made up of segments that are bounded by straight lines or by arcs of parabolas. By taking more arbitrary divisions in the length of a bar one obtains more accurate results in the general case. For most practical problems no more than five or six segments are necessary.

The procedure may be applied to other problems which depend on the same general principles. In mathematical terms, the procedure may be applied to the process of numerical integration of certain types of differential equations, in some cases directly, and in other cases by a sequence of successive approximations.

The essential features of the procedure are not new. The writer's first acquaintance with the concepts involved in this paper came some years ago from lectures in graduate courses at the University of Illinois, Urbana, Ill.,

Note.—Published in May, 1942, *Proceedings*.

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by Hardy Cross and H. M. Westergaard, Members, Am. Soc. C. E. Specific procedures have been discussed previously in engineering literature; for example, the application of similar graphical and analytical procedures to buckling of bars has been made by L. Vianello,[2] F. Engesser,[3] and others[4]; and the application of a graphical procedure to vibration of bars and shafts has been indicated by A. Stodola.[5,6] Methods of obtaining increased accuracy with a numerical procedure have been suggested by A. S. Niles,[7] Assoc. M. Am. Soc. C. E., and R. V. Southwell,[8,9] among others. The procedure suggested by Professor Southwell is in some respects similar to that described herein. However, the generalization of the procedure, the manner of application to specific problems, the treatment of functions with cusps or discontinuities, the simplified procedure for continuous functions with continuous derivatives, and the method of computing shears, slopes, or first derivatives, are essentially new, are more useful or more accurate than previous methods, and to the writer's knowledge have not been described previously.

Part I.—Computation of Moments in Beams

Introductory.—The calculation of the values of a function of a single variable, when the magnitude of the second derivative of the function is known, is a fundamental part of a group of physical problems, examples of which are the determination of the deflection of a string, or of a beam, and the computation of moments in a beam due to given loads. Analogies may be drawn between these and similar problems, since generally they may all be solved by the same procedures.

The method of computation described herein is a numerical procedure permitting as accurate a determination as is desired of the values of a function for specific values of the variable. The method is described in terms of calculation of moments in a beam for a given system of loads, but the application to other problems is also indicated, and particular application is made to the problem of buckling of bars.

Treatment of Concentrated Loads.—A fundamental part of the procedure depends on the rapid and systematic calculation of shears and moments in a beam subjected to a series of concentrated loads. Essentially, the process is to compute the shears from one end of the beam to the other by adding or sub-

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[2] "Graphische Untersuchung der Knickfestigkeit gerader Stäbe," by L. Vianello, *Zeitschrift des Vereins Deutscher Ingenieure*, Vol. 42, 1898, pp. 1436–1443.

[3] "Über die Knickfestigkeit von Stäben veränderlichen Trägheitsmomentes," by F. Engesser, *Zeitschrift des österreichischen Ingenieur- und architektenvereins*, Vol. 61, 1909, pp. 544–548.

[4] A discussion of some of these methods is given in "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 84–88 and 131–133.

[5] "Steam Turbines," by A. Stodola and L. C. Loewenstein, 2d Revised Ed., New York, N. Y., 1906, pp. 185–186.

[6] See also, for example, "Mechanical Vibrations," by J. P. Den Hartog, New York, N. Y., 1934, pp. 174–178.

[7] "Airplane Structures," by A. S. Niles and J. S. Newell, 2d Ed., New York, N. Y., 1938, Vol. I, pp. 133–136, and Vol. II, pp. 126–134.

[8] "Relaxation Methods Applied to Engineering Problems, I. The Deflexion of Beams Under Transverse Loading," by K. N. E. Bradfield and R. V. Southwell, *Proceedings*, Royal Soc. of London, Series A, Vol. 161, 1937, pp. 155–181, especially pp. 163–165.

[9] "Relaxation Methods Applied to a Spar of Varying Section, Deflected by Transverse Loading Combined with End Thrust or Tension," by R. J. Atkinson, K. N. E. Bradfield, and R. V. Southwell, *Reports and Memoranda No. 1822*, Aeronautical Research Committee, London, 1937.

tracting the successive loads, then to compute the moments by adding or subtracting the successive shears, multiplied by the length of beam over which the shear acts. The latter step is simpler if all the lengths between points of application of the concentrated loads are the same. However, the general case is not difficult, and the modification of the procedure described herein, to handle the general case, is obvious and will not be discussed.

To avoid confusion, a definite sign convention will be adopted in the work that follows. Moments will be considered positive when producing compression in the upper fibers of the beam. Shears will be considered positive when the resultant force to the left of a section is upward. Loads will be considered positive when the load acts upward. The latter convention is chosen so as to permit successive calculation of shears or moments always by adding, respectively, loads or shears, from left to right, and by subtracting the proper quantities from right to left.

When the shear or moment at any point is known the calculation can always be started from that point, but when only the moments at two points are known, the calculation of shears cannot be performed directly. However, a linear moment diagram, which corresponds to a constant shear, and therefore to no load, can always be added to the moments computed from some arbitrary shear chosen to start the calculations. Therefore, one may obtain the desired conditions relatively simply by merely adding a straight-line moment diagram as a correction, where it is needed.

The procedure is simplified by omitting the multiplication of the shears by the distance between loads until the end of the computations. That is, one can consider the loads as numerical quantities all multiplied by a common factor. The shears will be obtained from the loads, and will contain the same common factor. Then the moments will be computed as numerical quantities all multiplied by a common factor, which is the factor for the loads multiplied by the distance between loads.

The calculations are illustrated by the group of problems shown in Fig. 1. The units in which the loads are measured and the length of the panels are omitted purposely: These may have any values. The beam is divided into six equal segments, and the loads are shown in Fig. 1(*a*). The loading is the same for the different problems, but the manner of support and the method of performing the computations vary in the problems. In Fig. 1(*b*) the beam is cantilevered from the right end. Therefore at the left end the shear is zero and the moment is zero. In Figs. 1(*c*), 1(*d*), and 1(*e*) the beam is simply supported. In Fig. 1(*c*) are given linear correction moments which may be added to the moments in Fig. 1(*b*) to satisfy conditions of simple support; that is, zero moment at the two ends of the beam. The same result is obtained in Fig. 1(*d*), starting with the loads but choosing the shears so as to obtain the correct moments directly. In Fig. 1(*e*) the procedure is carried through in what might be a more usual calculation. One starts with a shear of five at the left end, merely as a guess. Then a proper correction to the moments is written in. The details of the calculations are self-explanatory.

Treatment of Distributed Loads.—When distributed loads are applied to the beam, one can choose equivalent concentrated loads that produce the same shears and moments at certain specified sections of the beam, and thereby handle the problem with the aforementioned single procedure. In so far as statics is concerned, the beam with the distributed load applied directly is equivalent to a system of simply supported stringers resting on the beam, and transmitting the distributed load to the beam as a series of concentrations which are the stringer reactions. The statical equivalence is illustrated in

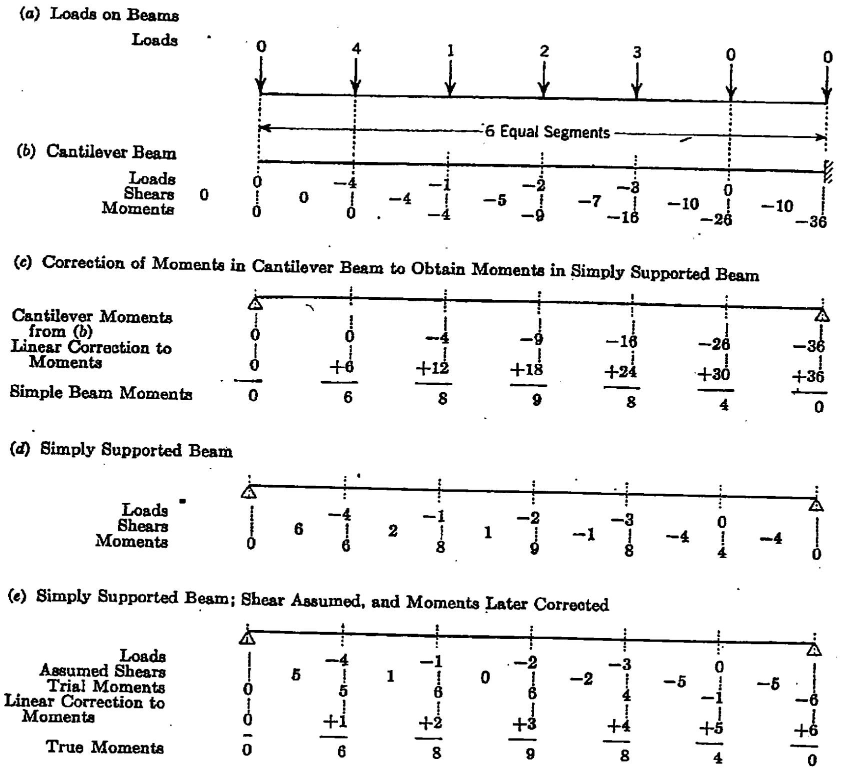


FIG. 1

Fig. 2. It may be observed that the moment or shear at any section of the original beam is equal to the moment or shear at any section through the beam and stringer of the beam-stringer substitute.

One obtains correct moments and, with some care in separating the two subreactions that make up the substitute concentrated load at a point, one obtains correct shears in the original beam at the points of support of the fictitious stringers, by considering a substitute structure loaded only by concentrated loads which are the reactions on the fictitious stringers. One also obtains correct reactions at the ends of the beam.

For a load diagram which consists of straight-line segments, the equivalent concentrated loads are readily determined directly. Formulas for the equivalent loads[10] are stated in Fig. 3. To illustrate the use of the procedure for such a load diagram, several simple problems are shown in Fig. 4. In Fig. 4(*a*), a uniform load on a simply supported beam is considered. Solutions are given in Fig. 4(*b*) for a triangular load diagram on a cantilever beam, and in Fig. 4(*c*)

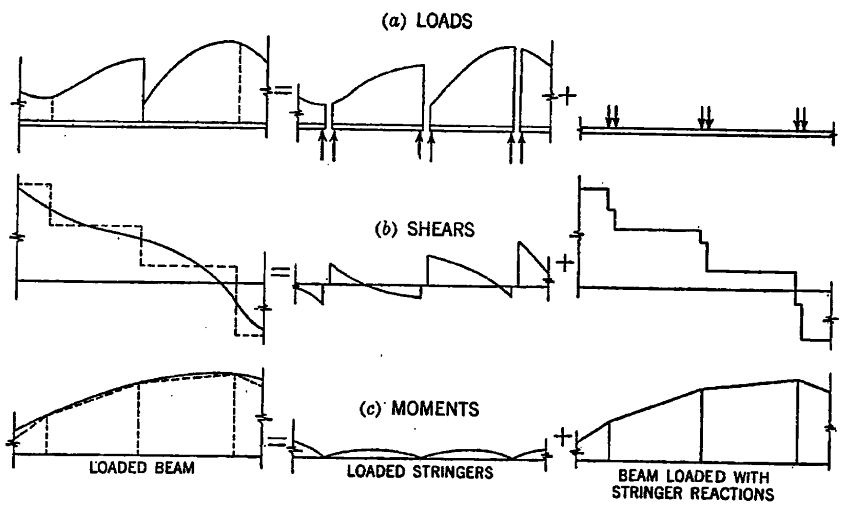


FIG. 2.—STATIC EQUIVALENCE OF BEAM WITH BEAM AND STRINGERS

for a triangular load diagram on a simple beam. In Figs. 4(*a*) and 4(*b*) the shears are computed at intermediate points; consequently the equivalent concentrations are shown in two parts. In Fig. 4(*c*) only the shears at the supports (that is, the reactions) are computed, and therefore only the total equivalent

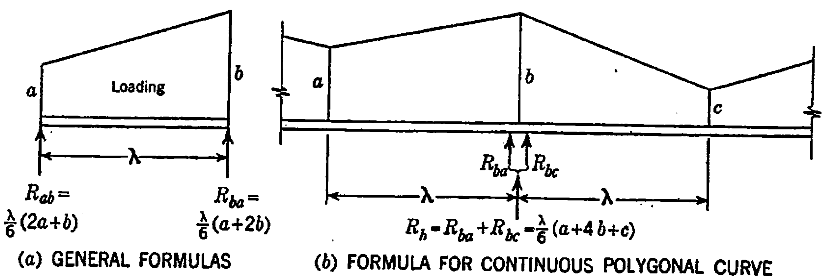


FIG. 3.—FORMULAS FOR EQUIVALENT CONCENTRATED LOADS, FOR POLYGONAL LOADING CURVE

loads are shown. Note that the moments in Fig. 4(*c*) could have been obtained from Fig. 4(*b*) by adding a linear moment diagram.

One can obtain formulas for more complicated types of load distribution. For practical purposes a distribution that varies according to the ordinates to

[10] The same formula for an equivalent concentration for a polygonal loading curve has been given in "Die graphische Statik der Baukonstruktionen," by H. Müller-Breslau, Vol. 2, Pt. II, 2d Ed., Leipzig, 1925, p. 44.

an arc of a second-degree parabola is sufficiently general, since it is possible to approximate almost any curve by segments of second-degree parabolas. Formulas for the equivalent concentrations for such a load are given in Fig. 5,

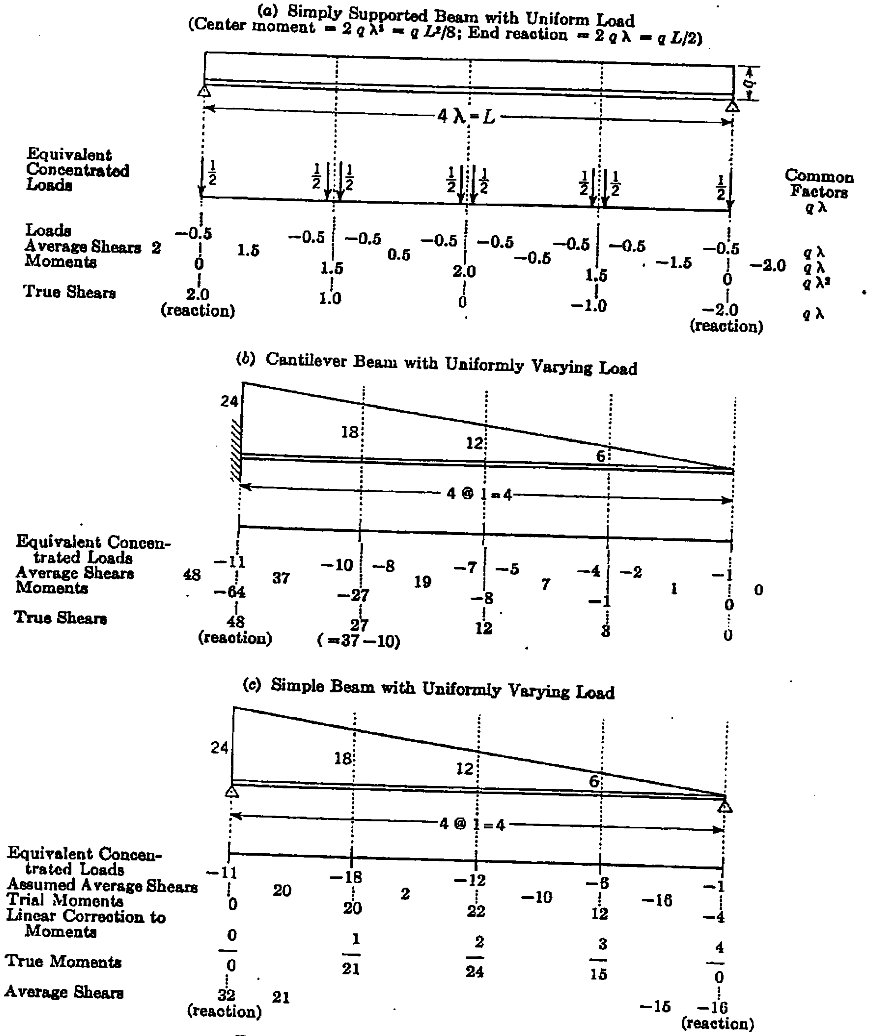


FIG. 4.—APPLICATION OF EQUIVALENT LOADS

in terms of three ordinates to the load distribution curve. The formula in Fig. 5(*b*) for a smooth loading curve was presented by A. Nádai in 1925[11]; and, what amounts to an equivalent formula for the smooth loading curve, with

[11] "Die elastischen Platten," by A. Nádai, Berlin, 1925, p. 209, Eq. 13.

additional terms, was derived by Professor Southwell in 1937.[8] A derivation of the formulas in Fig. 5 appears in the Appendix of this paper. It is noted that one of the three ordinates in Fig. 5(*a*) need not be an actual ordinate to the loading curve, but can be an extrapolated value. The formulas in Fig. 5 reduce to those in Fig. 3 when $a + c = 2\,b$, or when the parabola becomes a straight line. In general, the formulas in Fig. 5 may be used for any distributed loading to give a reasonably good approximation, and they are used in the remainder of this work.

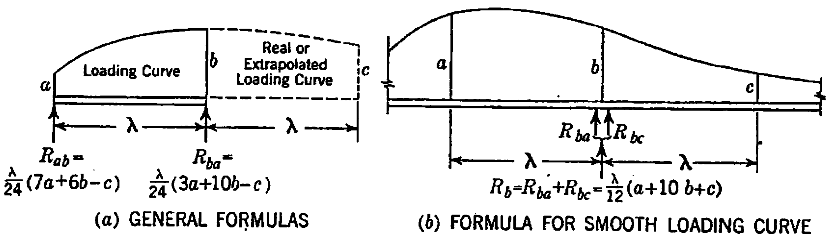


FIG. 5.—FORMULAS FOR EQUIVALENT CONCENTRATED LOADS

Problems in which use is made of the formulas in Fig. 5 are given in Part II. A simpler manner of using the results in Fig. 5 for curves that have no discontinuities, nor abrupt changes in slope, is also illustrated in the material that follows.

PART II.—CALCULATION OF DEFLECTION OF BEAMS

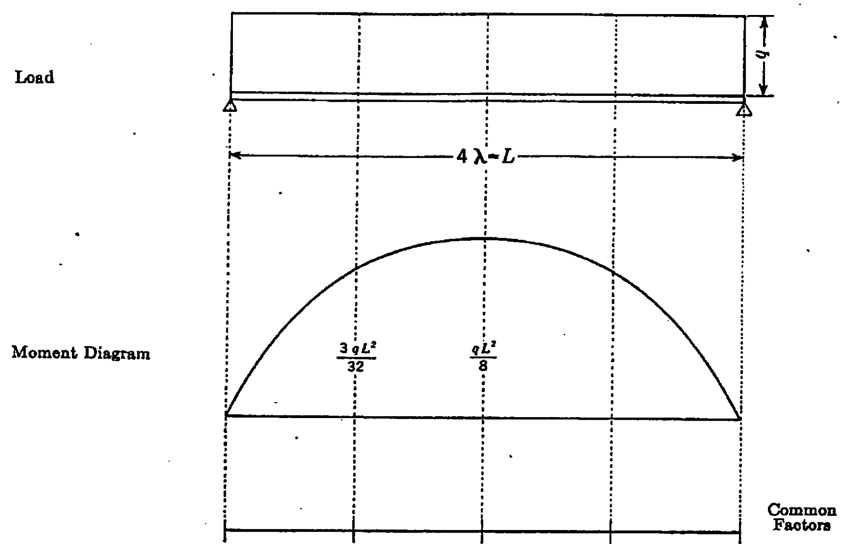
General Relations and Definitions.—A direct analogy can be drawn relating loads, shears, and moments in a beam to "angle changes," slopes, and deflections of a beam,[12] where the "angle changes" are the quantities giving the change of slope per unit length—that is, values of moment M divided by modulus of elasticity E and by moment of inertia I for an elastic beam with small deflections. The following sign convention is adopted in order that the analogy may hold without change of signs.

The "angle change" is defined as $-\dfrac{M}{E\,I}$; a positive "angle change" is considered as an upward load and therefore as a positive load. Then positive slope corresponds to an increase in deflection from left to right, and corresponds to a positive shear. Finally, positive deflection is taken as downward, and corresponds to a positive moment. A "concentrated angle change" corresponds to an abrupt change in slope at a point, and may be considered in the calculations without difficulty.

As a simple example of the use of the procedure, consider the deflection of a simply-supported beam of constant cross section subjected to uniform load, as in Fig. 6. The moment diagram is a parabola. Therefore the procedure will yield exact results with as many or as few segments as are desired. The calculations are shown for four segments in the length of the beam. The correct center deflection would have been obtained even if only two segments had been

[12] See, for example, "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y., 1932, pp. 28–30.

considered. Note that the constant factors in moment, angle changes, slopes, and deflections are written as multipliers at the right of the calculations. The equivalent concentrated angle changes at the ends of the beam need not be computed if only the deflections are desired.



										Common Factors
Ordinates to Moment Diagrams	0		3		4		3		0	$\frac{q L^2}{32}$
Ordinates to Angle-Change Diagrams	0		−3		−4		−3		0	$\frac{q L^2}{32 E I}$
Equivalent Concentrated Angle Changes	−7		−34		−46		−34		−7	$\frac{q L^2}{32 E I} \frac{\lambda}{12}$
Average Slope		57		23		−23		−57		$\frac{q L^2}{32 E I} \frac{\lambda}{12}$
Deflection	0		57		80		57		0	$\frac{q L^2}{32 E I} \frac{\lambda^2}{12}$

Center Deflection $= \frac{80 \, q L^2}{32 \, E I} \frac{\lambda^2}{12} = \frac{5 \, q L^4}{384 \, E I}$

Quarter Point Deflection $= \frac{57}{80}$ Center Deflection

Slope at End of Beam $= (57 + 7) \frac{q L^3}{32 \, E I} \frac{\lambda}{12} = \frac{q L^3}{24 \, E I}$

FIG. 6.—DEFLECTIONS FOR SIMPLY SUPPORTED BEAM WITH UNIFORM LOAD

Simplified Procedure for Smooth Angle-Change Curves.—It can be shown that, for the determination of deflections (or moments) alone, a simpler procedure may be used which avoids the calculation of the equivalent concentrated angle changes (or loads) from a distribution of angle changes (or loads) that has no discontinuities nor abrupt changes in slope in the region considered.

From the formula in Fig. 5(*b*) applying to a smooth curve, one has the relation:

$$R_b = \lambda \, b + \frac{\lambda}{12} (a - 2 \, b + c) \quad \text{.................. (1)}$$

Then one can consider the equivalent concentration at any point such as *b* as made up of two parts: (1) The ordinate to the curve at the point multiplied by λ; and (2) a correction which is $\frac{\lambda}{12} (a - 2 \, b + c)$. The correction loads at all the points, however, produce a deflection that is proportional to the original angle-change curve; actually, deflections at the various points due to the correction are $\frac{\lambda^2}{12}$ times the value of the distributed angle change at the point, plus any linear diagram required to satisfy the boundary conditions. This is obvious from the form of the equation. A proof is demonstrated in Fig. 7.

	← λ →		← λ →		← λ →		← λ →		Common Factors
Ordinates to the Angle-Change Diagram	*a*		*b*		*c*		*d*	*e*	
Part (2) of the Equivalent Concentrated Angle Change			$c - 2b + a$		$d - 2c + b$		$e - 2d + c$		λ/12
Assumed Average Slopes, Part (2)		$b - a$		$c - b$		$d - c$		$e - d$	λ/12
Deflections, Part (2), with Which Any Lineal Set of Deflections May Be Combined	*a*		*b*		*c*		*d*	*e*	λ²/12

FIG. 7.—DERIVATION OF PART (2) OF THE DEFLECTIONS FOR A SMOOTH CURVE OF ANGLE CHANGES

The problem of Fig. 6 is solved in Fig. 8 by use of the modified procedure. It is noted that the equivalent concentrated angle changes are not written; consequently the slopes must be computed from the original distributed angle changes multiplied by λ, as indicated in the factors to the right of the calculations. One should be careful that part (2) of the deflection is written with its proper sign. It is always $\frac{\lambda^2}{12}$ times the ordinates to the curve of distributed angle changes, and has the same sign as the distributed angle change.

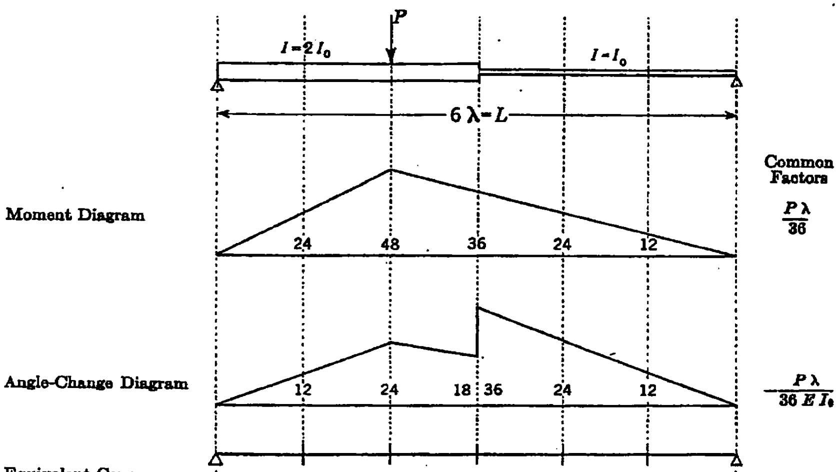
					← 4 λ = L →					
Ordinates to the Angle-Change Diagram	0		−3		−4		−3		0	$\frac{q L^2}{32 E I}$
Average Slope, Part (1)		5		2		−2		−5		$\frac{q L^2 \lambda}{32 E I}$
Deflections, Part (1)	0		5		7		5		0	$\frac{q L^2 \lambda^2}{32 E I}$
Deflections, Part (2)	0		$-\frac{3}{12}$		$-\frac{4}{12}$		$-\frac{3}{12}$		0	$\frac{q L^2 \lambda^2}{32 E I}$
Total Deflection	0		$\frac{57}{12}$		$\frac{80}{12}$		$\frac{57}{12}$		0	$\frac{q L^2 \lambda^2}{32 E I}$

FIG. 8.—ALTERNATIVE PROCEDURE FOR PROBLEM OF FIG. 6

When the original angle-change diagram is linear (that is, either constant or uniformly varying) it is unnecessary to consider part (2) of the deflections since one may add a linear set of deflections to make the net effect of $\frac{\lambda^2}{12}$ times the original angle changes and the added linear deflection zero. Then one may add whatever other linear deflections are required to satisfy the conditions of the problem.

Where there is a break in the curve or a discontinuity in slope, it is impossible to use this simplified procedure without modification. To avoid confusion the general procedure is recommended for such problems.

Further examples of the use of the general procedure and of the modified procedure are given in Figs. 9 and 10, which illustrate respectively the calcula-



														Common Factors
Equivalent Concentrated Angle Changes			−12		−21		−26		−24		−12			$\frac{P\lambda^2}{36EI_0}$
Assumed Average Slopes		33		21		0		−26		−50		−62		$\frac{P\lambda^2}{36EI_0}$
Trial Deflections	0		33		54		54		28		−22		−84	$\frac{P\lambda^3}{36EI_0}$
Linear Correction to Deflections	0		14		28		42		56		70		84	$\frac{P\lambda^3}{36EI_0}$
True Deflections	0		47		82		96		84		48		0	$\frac{P\lambda^3}{36EI_0}$

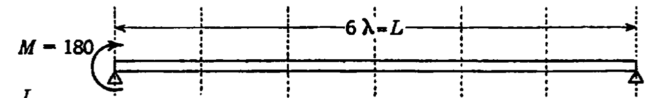
FIG. 9.—DEFLECTION OF BEAM WITH CHANGE IN SECTION

tion of deflections for a member with an abrupt change in section, and for a member of varying cross section. The deflections in Fig. 9 will be exact since the $\frac{M}{EI}$-curve is composed of straight-line segments. However, the deflections in Fig. 10 are not exact since the curve of angle changes is not composed of straight-line or parabolic segments. More nearly correct results are obtained by taking more divisions in the length of the beam. The number of divisions actually taken (six) will yield results that are very accurate as is shown by comparison with a solution having twice the number of divisions, in Fig. 10(b), and with an "exact" solution in Fig. 10(c).

Analyses of Statically Indeterminate Beams.—By superposing the effects of different end moments one can solve the problem of a statically indeterminate beam also. For example, in Fig. 11(a) is shown the same beam as in Fig. 10, with a moment applied to the opposite end. The end slopes due to the mo-

ments applied in Figs. 10 and 11 are easily computed and are indicated on the figures. The calculation of end slope from the equivalent concentrated angle change at the end leads to a much greater accuracy than is possible by other means—for example, by methods involving differences of various orders of the final deflections only, as suggested by Professor Southwell.[13]

(a) Deflection of Beam of Variable Section Subjected to Moment at Left End



$\frac{I}{I_0}$	1		2		3		4		5		6		7	Common Factors
Ordinates to Moment Diagram	180		150		120		90		60		30		0	
Ordinates to Angle-Change Diagram	−180		−75		−40		−22.5		−12		−5		0	$1/EI_0$
Assumed Average Slope, Part (1)		100		25		−15		−37.5		−49.5		−54.5		λ/EI_0
Trial Deflections, Part (1)	31.5		131.5		156.5		141.5		104.0		54.5		0	λ^2/EI_0
Deflections, Part (2)	−15.0		−6.3		−3.3		−1.9		−1.0		−0.4		0	λ^2/EI_0
Linear Correction to Deflections	−16.5		−13.8		−11.0		−8.3		−5.5		−2.8		0	λ^2/EI_0
Final Deflection	0		111.4		142.2		131.3		97.5		51.3		0	λ^2/EI_0
End Slopes:														
From Average Slope	111.4												−51.3	λ/EI_0
From Equivalent Concentrated Angle Change at End	69.6												−0.8	λ/EI_0
Total End Slope	181.0												−52.1	λ/EI_0

(b) Results of Solution with 12 Divisions of Length $\frac{\lambda}{2}$:
(values given only for alternate points corresponding to those above)

Final Deflections	0	110.7	141.4	130.8	97.1	51.1	0	λ^2/EI_0
End Slopes	177.4						−51.9	λ/EI_0

(c) Exact Solution by Integration

Deflections	0	110.63	141.37	130.76	97.08	51.12	0	λ^2/EI_0
End Slopes	176.75						−51.89	λ/EI_0

FIG. 10.—DEFLECTION OF BEAM OF VARIABLE SECTION BY MODIFIED PROCEDURE

The end slopes in Figs. 10 and 11 differ slightly from the exact values obtained by integration. A much better agreement with the exact values is obtained if a greater number of segments in the length of the beam are used. There is a rapid change in the values of the angle-change curve at the left end of the beam in Figs. 10 and 11, and consequently a greater error in the slopes for this end than for the right end, by the approximate procedure. It should also be pointed out that the slope at the right end in Fig. 10(a) should be equal to the slope at the left end in Fig. 11(a), by Maxwell's theorem of reciprocal deflections. The difference is due to the fact that the procedure involves

[13] "Relaxation Methods Applied to Engineering Problems, I. The Deflexion of Beams Under Transverse Loading," by K. N. E. Bradfield and R. V. Southwell, *Proceedings*, Royal Soc. of London, Series A, Vol. 161, 1937, pp. 166–167.

some slight inaccuracies, which amount to analyzing slightly different structures in the two cases.

From the moments and slopes in Figs. 10 and 11, one can find, for example, the stiffness and the carry-over factor for the left end of the beam by adjusting the moments at the ends to give the proper conditions as shown in Fig. 11(*c*).

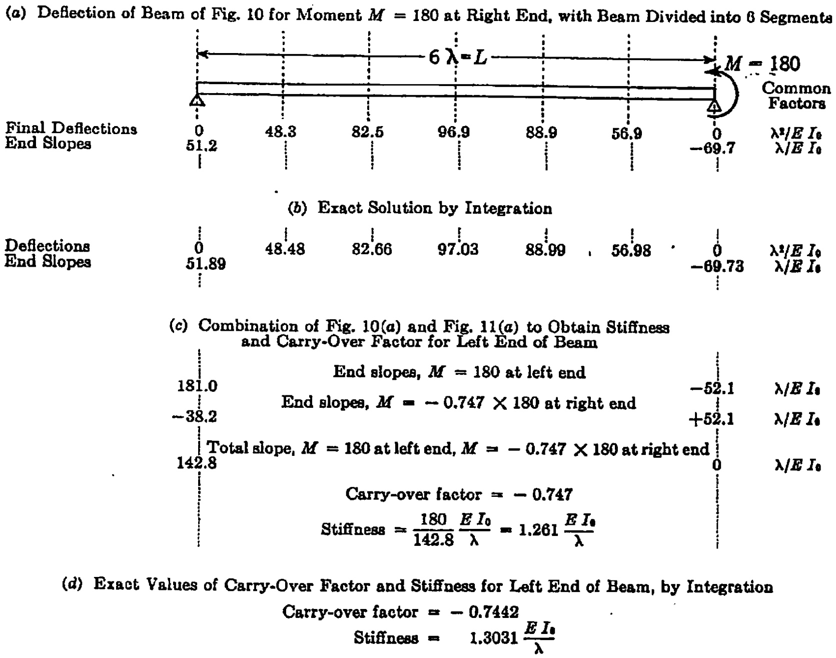


FIG. 11.—CALCULATION OF STIFFNESS AND CARRY-OVER FACTOR

For comparison, "exact" values of stiffness and carry-over factor are shown in Fig. 11(*d*), obtained by integration. The agreement is close although only six segments were used in the approximate procedure.

PART III.—DEFLECTION OF BEAMS WITH AXIAL LOADS; BUCKLING OF COLUMNS

General Procedure.—With an accurate procedure available for computing deflections of a beam when the moments are known, it is possible to set up a relatively simple procedure for handling deflections of bars subjected to axial loads as well as lateral loads, by successive approximations. In so far as the final deflections are concerned, the effect of lateral loads on the bar is the same as the effect of initial deflection of the bar from a straight line.

The following method of analysis is recommended for the general case:

(1) Divide the bar into a number of segments. Compute the deflections of the bar due to the lateral loads only, and add these deflections to the initial deviations from a straight line. Let the total deflection with no axial loads be denoted by the symbol w_i.

(2) Guess at an assumed additional deflection, w_a, which is to represent the effect of the axial forces on the bar. Let the sum of w_a and w_i be denoted by w_0; that is,

$$w_0 = w_i + w_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(3) Compute the moments due to the axial loads on the bar, corresponding to the deflections w_0.

(4) Determine the deflections of the bar for the moments computed in step (3). Let these deflections be denoted by w'_a.

(5) Compare w'_a and w_a. If they are equal, w_a is the correct additional deflection of the bar, and w_0 is the correct total deflection of the bar. If they are not equal, repeat steps (2) to (5) until a desired measure of agreement is reached. One may take the values of w'_a in step (4) as a new set of assumed values of w_a, or one may modify these values in order to hasten the process and obtain agreement more rapidly between the assumed deflections and the resulting deflections.

It is necessary to point out that the procedure will work to advantage only when w'_a is a better approximation to the true additional deflections than w_a; in other words, the procedure works best when the sequence of successive approximations converges. It may not work at all when the sequence diverges or oscillates. One can formulate conditions that will insure convergence; but for practical purposes it will be evident that one either approaches a definite result or does not; and if the calculations approach a definite answer, it is the correct answer. Various "tricks" are possible in solving problems in which convergence is slow, or in which there is actually divergence of the results. However, such problems are not common. The writer does not wish to confuse this presentation with too elaborate a set of procedures for exceptional cases. It is sufficient to point out that if by any means w'_a and w_a can be made equal at all division points, one has the correct deflections. By trial, or by a systematic procedure, or by use of simultaneous equations, the two sets of values can always be made equal (even when the routine procedure of using the results in (4) as a new step (2) diverges), since one may take any arbitrary set of values of w_a.

Examples of the general procedure are given subsequently herein. Usually it is possible to obtain a good set of values of w_a if one has available a solution of the problem of pure buckling of the particular bar considered. For this reason a discussion of pure buckling will be given first.

Treatment of the Problem of Pure Buckling Without Lateral Loads.—Consider a bar subjected only to axial loads, without lateral load or initial deflection. Then the quantities w_i in step (1) of the general procedure are zero. The axial loads are to be determined so that an assumed set of deflections w_a corresponds to the same set of deflections w'_a, which means that the deflected bar is in a position of neutral equilibrium, and is on the point of reaching a position of stable (or possibly unstable) equilibrium which is different from the original undeflected position.

The special procedure becomes:

(a) Guess at a set of deflections w_a.

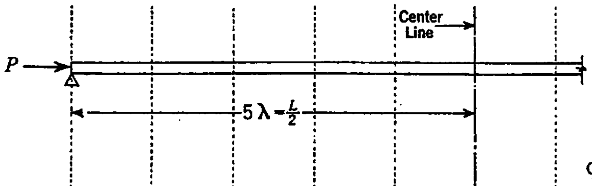
(b) Compute moments corresponding to the deflections w_a and an assumed set of values of the axial loads. It is convenient to consider generalized loads. That is, one considers the symbol P to represent all of the axial loads, or the system of such loads, acting on the bar. By assigning values to P, one assigns values to each of the individual loads which P represents.

(c) From the moments in step (b), compute deflections w'_a.

(d) Compare the deflections w'_a and w_a. If they are proportional—that is, if they can be made identically equal for a particular value of P, or for a particular set of values of the axial loads—there is a critical buckling load and the configuration of the bar corresponds to that load.

Again, certain questions may be raised regarding the convergence of a sequence of computations,[14] but these are beyond the scope of this paper; moreover, for most practical problems the difficulties do not arise.

Consider, for example, the calculation of the critical buckling load for a simply supported bar of constant cross section loaded at the ends with axial compressive forces. The calculations for an assumed parabolic deflection curve, symmetrical about the center line, are shown in Fig. 12. Only half the



														Common Factors
Assumed Deflections, w_a	0		36		64		84		96		100[a]		96	
Distributed Angle Changes	0		−36		−64		−84		−96		−100		−96	P/EI
Average Slope, Part (1)		330		294		230		146		50		−50		$P\lambda/EI$
Deflections, Part (1)	0		330		624		854		1,000		1,050		1,000	$P\lambda^2/EI$
Deflections, Part (2)	0		−3.0		−5.3		−7.0		−8.0		−8.3		−8.0	$P\lambda^2/EI$
Resultant Deflections, w'_a	0		327.0		618.7		847.0		992.0		1,041.7		992.0	$P\lambda^2/EI$
Ratio, $\frac{w_a}{w'_a}$	(0.1200)[b]		0.1101		0.1034		0.0992		0.0968		0.0960		0.0968	$EI/P\lambda^2$

(a) Average ratio, neglecting ratio for ends $= 0.1017 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 10.17 \frac{EI}{L^2}$

(b) Ratio of sums of deflections, $\frac{\Sigma w_a}{\Sigma w'_a} = 0.0998 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 9.98 \frac{EI}{L^2}$

(c) Best ratio by least squares, $\frac{\Sigma w_a w'_a}{\Sigma w'_a w'_a} = 0.0987 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 9.87 \frac{EI}{L^2}$

[a] Beam and deflections symmetrical about center line. [b] Ratios of end slopes.

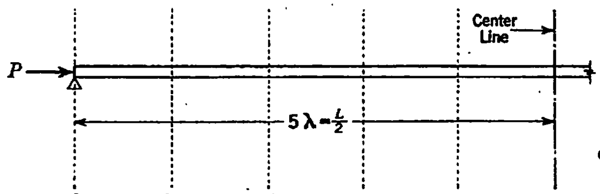
FIG. 12.—CRITICAL BUCKLING LOAD FOR BAR OF CONSTANT SECTION, STARTING WITH ASSUMED PARABOLIC DEFLECTION CURVE

bar is considered since the structure and the deflections are symmetrical. It is seen that the ratio of w_a to w'_a is not constant; the different values of this ratio are recorded, and give the value of P required to produce equality of deflections at the particular points. A repetition of the calculation with new

[14] See, for example, "Zur Konvergenz des Engesser-Vianello-Verfahrens," by A. Schleusner, Berlin, 1938.

values of w_a, equal to, or proportional to, the values of w'_a shown in Fig. 12, would give more nearly uniform ratios. The best value of the critical load may be taken as the average of the ratios, or as some weighted average; in Fig. 12, three different values of the critical load, computed in different ways, are reported. From similar calculations, it is the writer's conclusion that in most cases a reasonably good approximation to the critical load is the ratio of the sums of the ordinates to the curves of w_a and w'_a.

A more uniform set of ratios with a correspondingly better approximation to the critical load is given with a curve that more nearly approaches the true buckling configuration. In Fig. 13 a set of values is assumed for w_a approxi-



												Common Factors
Assumed Deflections, w_a	0		31		59		81		95		100	
Distributed Angle Changes	0		−31		−59		−81		−95		−100	P/EI
Average Slope, Part (1)		316		285		226		145		50		$P\lambda/EI$
Deflection, Part (1)	0		316		601		827		972		1,022	$P\lambda^2/EI$
Deflection, Part (2)	0		−2.6		−4.9		−6.8		−7.9		−8.3	$P\lambda^2/EI$
Resultant Deflection, w'_a	0		313.4		596.1		820.2		964.1		1,013.7	$P\lambda^2/EI$
Ratio, $\frac{w_a}{w'_a}$	..		0.0989		0.0990		0.0987		0.0985		0.0986	$EI/P\lambda^2$

Average ratio $= 0.0988 \frac{EI}{P\lambda^2}$

Ratio of sums $= 0.0987 \frac{EI}{P\lambda^2}$

$\therefore\ P_{cr} = 0.0987 \frac{EI}{\lambda^2} = 9.87 \frac{EI}{L^2}$

FIG. 13.—CRITICAL BUCKLING LOAD FOR BAR OF CONSTANT SECTION

mately proportional to the values of w'_a determined in Fig. 12. The result is practically exact. In both Figs. 12 and 13, the true value of the critical load is $\pi^2 EI/L^2$; or, $9.870\, EI/L^2$.

Obviously it is possible to find different patterns of deflections corresponding to different values of critical loads for the same bar. In general, only the lowest critical load is of significance as far as pure buckling is concerned, since the higher loads must correspond to essentially unstable positions of equilibrium; but, if an initial deflection curve is assumed that contains no components of the configuration corresponding to the lowest critical buckling load, the lowest load cannot be obtained from this procedure (nor would it be obtained from any other available procedure, such as methods involving minimum of energy). Such a situation would follow from the assumption of a deflection curve anti-symmetrical about the center line for the beam in Fig. 12. One would reject such a curve intuitively for this problem. Yet in an unusual case, it might be possible that a designer may reject, unthinkingly, the configuration that corresponds to the lowest buckling load. An example of such a case is shown subsequently in Fig. 20.

Ordinarily, convergence of several different sequences of computations involving different shapes of assumed deflection curves, to the same final shape, would be a sufficient indication that the designer had reached the configuration corresponding to the lowest critical load. In some cases, however, the convergence of a sequence of computations may be very slow; this will be so when the next higher critical load differs only slightly from the lowest critical load. Methods of handling such problems can be derived but are beyond the scope of the present paper.

Determination of Maximum and Minimum Values for the Critical Load.—In general, the lowest critical buckling load must have a value between the limits defined by the smallest and largest values of the ratio of w_a to w'_a, when all values of w_a and w'_a are positive. One can reason as follows to justify this rule: If every point on the derived deflection curve lies outside of every point on the assumed deflection curve, the load must be greater than the load required to produce neutral equilibrium, since the bar is tending to deflect even farther away from its original straight configuration than assumed. This means that the initial straight configuration is now an unstable position of equilibrium. On the other hand, if every point on the derived deflection curve lies between the original straight configuration and the assumed configuration, then the load must be less than the load required to produce neutral equilibrium. Evidently, in this case, the undeflected position is a position of stable equilibrium; but the two conditions described correspond to the maximum and the minimum values of the ratio of w_a to w'_a. Therefore the critical buckling load must be between these limits. The rule is important for practical purposes; the designer can readily detect between what limits the buckling load must lie.

In using this rule to bound the value of the critical load, it must be remembered that the structure set up for analysis is not exactly the same as the structure it represents, although with a reasonably large number of divisions the two are closely similar. The process of dividing the bar into segments is equivalent to substituting for it a slightly different structure. This becomes evident if the buckling load is computed for a bar divided into only two segments, as in Fig. 14(*a*).

In certain cases the foregoing rule is inapplicable. Care must be taken in using it when axial loads are applied other than at the ends of a bar. Also, the rule would be misleading in such cases where the lowest critical load corresponds to a deflection curve that has both positive and negative deflections, whereas the next higher critical load might correspond to a deflection curve with only positive ordinates.

Illustrative Problems for Pure Buckling.—The problems shown in Figs. 14, 15, and 16 illustrate further uses of the procedure for computing the critical load for a beam subjected to pure buckling.

The effect of taking different numbers of segments in the length of the bar is illustrated in Fig. 14, for a simply supported bar of uniform section subjected to end thrust. The error in the buckling load computed by the approximate procedure described herein, compared with the exact buckling load, is 2.74% for two segments, 0.52% for three segments, and 0.16% for four segments, in the full length of the bar.

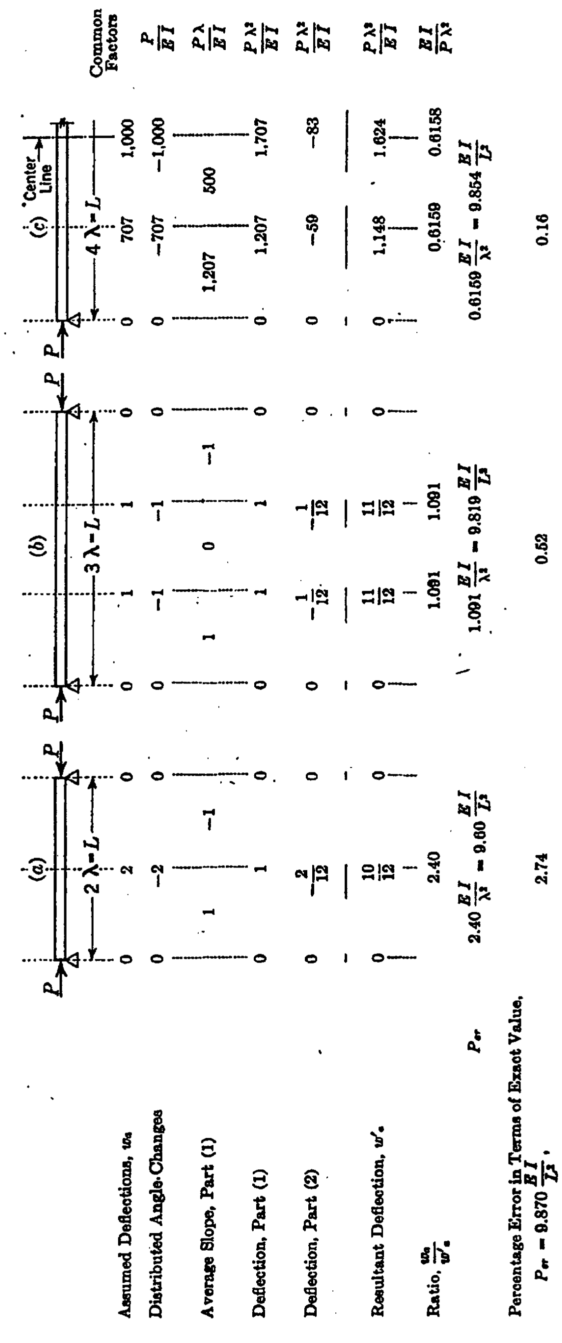
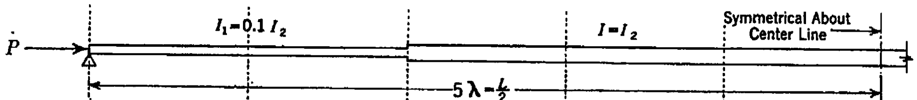


FIG. 14.—EFFECT ON BUCKLING LOAD OF TAKING DIFFERENT NUMBERS OF SEGMENTS FOR BAR OF CONSTANT CROSS SECTION



(a)

												Common Factors
Assumed Deflection, w_a	0		51.35		80.36		91.10		97.75		100.00	
Distributed Angle Changes	0		−513.5		−803.6 ¦ −80.36		−91.10		−97.75		−100.00	$P/E I_2$
Equivalent Concentrated Angle Changes			−494.88		−404.90		−90.76		−97.38		−99.63	$P\lambda/E I_2$
Average Slope		1,137.74		642.86		237.96		147.20		49.82		$P\lambda/E I_2$
Resultant Deflection, w'_a	0		1,137.74		1,780.60		2,018.56		2,165.76		2,215.58	$P\lambda^2/E I_2$
Ratio, $\frac{w_a}{w'_a}$			0.04513		0.04513		0.04513		0.04513		0.04513	$E I_2/P\lambda^2$

$$P_{cr} = 0.04513\frac{E I_2}{\lambda^2} = 4.513\frac{E I_2}{L^2}$$

(b)

												Common Factors
Assumed Deflection, w_a	0		36		64		84		96		100	
Distributed Angle Changes	0		−360		−640 ¦ −64		−84		−96		−100	$P/E I_2$
Equivalent Concentrated Angle Changes			−353.3		−312.3		−83.3		−95.3		−99.3	$P\lambda/E I_2$
Average Slopes		893.9		540.6		228.3		145.0		49.7		$P\lambda/E I_2$
Resultant Deflection, w'_a	0		893.9		1,434.5		1,662.8		1,807.8		1,857.5	$P\lambda^2/E I_2$
Ratio, $\frac{w_a}{w'_a}$			0.0403		0.0446		0.0505		0.0531		0.0538	$E I_2/P\lambda^2$

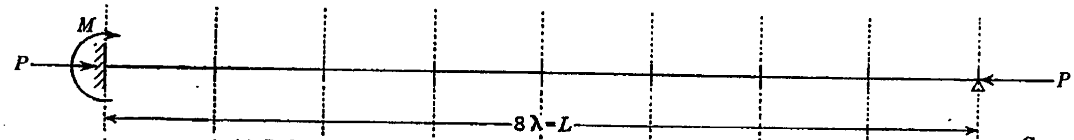
$$\sim P_{cr} = \frac{\Sigma w_a}{\Sigma w'_a} = 4.91\frac{E I_2}{L^2}$$

(c)

							Common Factors
Assumed Deflection, w_a	0	50	80	91	97	100	
Resultant Deflection, w'_a	0	1,120.8	1,758.3	1,995.5	2,142.1	2,191.9	$P\lambda^2/E I_2$
Ratio, $\frac{w_a}{w'_a}$		0.0446	0.0455	0.0456	0.0453	0.0456	$E I_2/P\lambda^2$

$$\sim P_{cr} = \frac{\Sigma w_a}{\Sigma w'_a} = 4.54\frac{E I_2}{L^2}$$

FIG. 15.—BUCKLING OF A BAR WITH CHANGE IN SECTION



(a) Deflection Due to Moment $M = 240$ at Left End

																		Common Factors
Moments	240		210		180		150		120		90		60		30		0	
Distributed Angle Changes	−240		−210		−180		−150		−120		−90		−60		−30		0	$1/E I$
Assumed Average Slope, Part (1)		500		290		110		−40		−160		−250		−310		−340		$\lambda/E I$
Trial Deflection, Parts (1) and (2)	0		500		790		900		860		700		450		140		−200	$\lambda^2/E I$
Linear Correction to Deflection	0		25		50		75		100		125		150		175		200	$\lambda^2/E I$
Resultant Deflection, w_a	0		525		840		975		960		825		600		315		0	$\lambda^2/E I$

Slope at left end $= (525 + 115)\frac{\lambda}{E I} = 640\frac{\lambda}{E I}$

(b) Deflections Due to End Loads with Assumed Deflections

																		Common Factors
Assumed Deflection, w_a	0		32		115		212		286		307		259		148		0	a
Distributed Angle Changes	0		−32		−115		−212		−286		−307		−259		−148		0	$P\,a/E I$
Assumed Average Slope, Part (1)		585		553		438		226		−60		−367		−626		−774		$P\lambda\,a/E I$
Trial Deflection, Part (1)	0		585		1,138		1,576		1,802		1,742		1,375		749		−25	$P\lambda^2 a/E I$
Deflection, Part (2)	0		−3		−10		−18		−24		−26		−22		−12		0	$P\lambda^2 a/E I$
Linear Correction to Deflection	0		3		6		9		13		16		19		22		25	$P\lambda^2 a/E I$
Intermediate Deflection, w_b	0		585		1,134		1,567		1,791		1,732		1,372		759		0	$P\lambda^2 a/E I$

End slope $= (585 + 3)\frac{P\lambda a}{E I} = 588\frac{P\lambda a}{E I}$

(c) Effect of Making Slope at Left End Zero, Moment at Left End $= -\frac{588\frac{P\lambda a}{E I}}{640\frac{\lambda}{E I}}(240) = -220.5\,P\,a$

										Common Factors
$-\frac{220.5\,P\,a}{240}w_a$	0	−482	−772	−896	−882	−758	−551	−289	0	$P\lambda^2 a/E I$
Resultant Deflection, w'_a	0	103	362	671	909	974	821	470	0	$P\lambda^2 a/E I$
Ratio, $\frac{w_a}{w'_a}$		0.310	0.318	0.316	0.314	0.315	0.315	0.315		$E I/P\lambda^2$

$$P_{cr} = 0.315\frac{E I}{\lambda^2} = 20.16\frac{E I}{L^2}$$

FIG. 16.—BUCKLING OF BAR FIXED AT ONE END

In Fig. 15(a), a solution is given for the buckling of a bar consisting of parts of constant but different moment of inertia. Because of the abrupt change in moment of inertia there is a discontinuity in the values of the angle changes in the bar. The result obtained with only five divisions in the half-length of the bar is

$$P_{cr} = 4.51 \frac{E I_2}{L^2} \qquad (3)$$

which compares with $4.50 \frac{E I_2}{L^2}$ given by Professor Timoshenko[15] as the "exact" value of the critical load. It should be remembered that several trials were necessary before as uniform a set of ratios as is shown in Fig. 15(a) was obtained; but the intermediate work can be done without refinements and the final result obtained fairly rapidly. For practical purposes it would not be necessary to go so far. For example, Figs. 15(b) and 15(c) might contain all the calculations required in most cases, where even the first step, starting with an assumed parabolic deflection curve, would be adequate for almost any practical problem.

In a similar manner, other problems involving variations in moment of inertia along the length of the bar may be solved. Where the variation is smooth (that is, without abrupt changes) the relatively simple modified procedure which does not require calculation of "equivalent" concentrated angle changes may be used.

The solution of the problem of buckling of a bar fixed at one end and simply supported at the other is shown in Fig. 16. The problem is solved by adding to a simply supported bar an end moment to annul the rotation at one end of the bar. The problem might also have been solved by dealing with a cantilever beam acted on by a direct thrust, and adding the effect of a lateral load at the end in order to make the deflection at the end zero. The results would have been exactly the same.

The procedure used in Fig. 16 may be outlined as follows:

(a) Find the deflections and end rotation of a simply supported bar due to a moment applied at one end. Denote the deflections by w_s.

(b) Assume a deflection curve for the bar fixed at one end and simply supported at the other. Denote the deflections by w_a. Compute the moments in the bar due to the direct loads and the deflections w_a. One may also include assumed moments to account in some measure for the effect of fixing the one end of the bar. In general, it would be desirable to include such "indeterminate" moments, although in Fig. 16 they were omitted.

(c) Compute the deflections w_b and the end rotation corresponding to the moments in step (b). If the end rotation is not zero, add such a moment as would be required to make it zero. This involves adding deflections also, proportional to w_s. Denote the resultant deflections by the symbol w'_a.

(d) Compare w_a and w'_a, as in the procedure described previously for determining buckling loads for statically determinate bars. If w_a and w'_a are similar, one has the correct shape of the deflection curve, and one can obtain the critical load. If w_a and w'_a are not similar, one may repeat steps (b) to (d) as many times as necessary, until one obtains a sufficiently good value of the critical load.

[15] "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 128–131.

A procedure similar to the foregoing may be developed for other statically indeterminate beams or columns.

Note that in Fig. 16(a) the moment diagram and the angle-change diagram are linear; therefore it was not necessary to compute part (2) of the deflections, as explained in section II of this paper. In Fig. 16(b) a common factor a is indicated for the deflections in order to make it clear that the end moment in Fig. 16(c) depends on the deflections. The final value of the critical load is practically exact.[16]

Illustrative Problems, Combined Axial and Lateral Loads.—When lateral loads act on a beam together with an end thrust, the effect of the end thrust is to produce additional deflections and additional moments beyond those produced by the lateral loads alone. The additional deflections are governed by the deflection due to the lateral load alone, and by the ratio of the axial loads to the critical value of the axial loads.

For the first step in the general procedure of solving such problems it is necessary to assume a set of values of the additional deflection, w_a. As a convenient approximation for the first trial value of w_a it is desirable to take w_a as follows:

$$w_a = \frac{1}{\frac{P_{cr}}{P} - 1} w_i \qquad (4)$$

in which P_{cr} is the magnitude of the critical buckling load, P is the magnitude of the actual load, and w_i is the sum of the initial deflection and the deflection due to the lateral load alone. When w_i is of the same shape as the deflection curve corresponding to the lowest critical buckling load, the value of w_a given by Eq. 4 will be exact.[17] In other cases, it hastens the convergence toward the correct value of w_a if w_a is assumed as suggested.

The calculations for a simply supported bar subjected to end thrusts and uniform lateral load are shown in Fig. 17. The values of w_i for the uniform load are computed first. The value of P_{cr} for the bar can be taken from Fig. 12. Then with the given load, $P = 0.02 \frac{E I}{\lambda^2}$, and the critical load, $P_{cr} = 0.0987 \frac{E I}{\lambda^2}$, one finds from Eq. 4 the following result:

$$w_a = 0.254\, w_i \qquad (5)$$

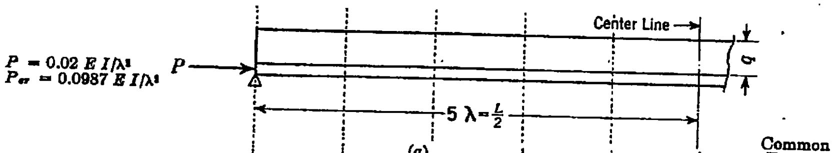
With this value of w_a, the computations in Fig. 17(b) lead to a set of values of w'_a which are practically equal to those assumed. If further refinement is desired one can repeat the calculations. One may also deal with additions to the values of w_a already assumed and obtain additions or corrections to w'_a; but in this problem no further computations appear to be necessary, and one may conclude that under the given conditions the effect of the axial load is to cause an apparent increase in the maximum moment due to the lateral loads alone of about 26%.

[16] "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 88–89.

[17] See, for example, "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), pp. 576–676, especially pp. 618–619. Note difference in notation, however.

Similar calculations are shown in Fig. 18 for a bar subjected to a moment at one end combined with direct thrust. Here the first approximation is not nearly so close to the final answer since the deflection curve due to the moment alone differs considerably from the configuration corresponding to the critical buckling load.

The procedure described here is applicable also to problems in which P is negative; that is, where axial tensions instead of compressions act on the bar. In such cases, however, the effect of the end tension is generally to reduce the deflections due to the lateral loads or initial eccentricities only. The same



(a)

	0		1		2		3		4		5	Common Factors
Moments Due to Lateral Load	0		36		64		84		96		100	$q\lambda^2/8$
Distributed Angle Changes	0		−36		−64		−84		−96		−100	$q\lambda^2/8EI$
Average Slope, Part (1)		330		294		230		146		50		$q\lambda^3/8EI$
Deflection, Part (1)	0		330		624		854		1,000		1,050	$q\lambda^4/8EI$
Deflection, Part (2)	0		−3		−5		−7		−8		−8	$q\lambda^4/8EI$
Deflection, w_i	0		327		619		847		992		1,042	$q\lambda^4/8EI$

(b)

	0		1		2		3		4		5	Common Factors
Assumed Deflection, w_a from Eq. 4	0		83		157		216		252		265	$q\lambda^4/8EI$
$w_0 = w_a + w_i$	0		410		776		1,063		1,244		1,307	$q\lambda^4/8EI$
Moments Due to P	0		8.20		15.52		21.26		24.88		26.14	$q\lambda^2/8$
Distributed Angle Changes	0		−8.20		−15.52		−21.26		−24.88		−26.14	$q\lambda^2/8EI$
Average Slope, Part (1)		82.93		74.73		59.21		37.95		13.07		$q\lambda^3/8EI$
Deflection, Part (1)	0		82.93		157.66		216.87		254.82		267.89	$q\lambda^4/8EI$
Deflection, Part (2)	0		−0.68		−1.29		−1.77		−2.07		−2.18	$q\lambda^4/8EI$
Resultant Deflection, w'_a	0		82		156		215		253		266	$q\lambda^4/8EI$

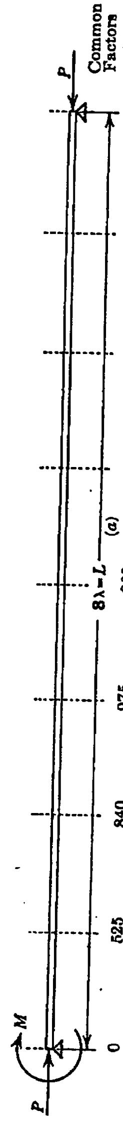
FIG. 17.—DEFLECTION OF A BAR SUBJECTED TO UNIFORM LOAD AND END THRUST

general procedure may be used. The value of w_a suggested in Eq. 4 will be negative, since, if the axial tensions are denoted by T, one has the result

$$w_a = -\frac{1}{\frac{P_{cr}}{T} + 1} w_i \qquad (6)$$

A difficulty arises in problems where T is numerically greater than the value of the lowest critical buckling load. In such cases the sequence of computations will oscillate, and will not converge. In general, each assumed value of w_a will lead to a value of w'_a which will be farther from the true configuration than w_a if w_a is not correctly chosen equal to its true value. Methods of solving such problems can be developed, however, and in general one can arrive eventually at reasonably good results since the effect of the end tensions can never be to produce greater deflections than w_i except at a few points. Further discussion of problems such as these will not be given in the present paper.

Buckling Due to Axial Loads Applied at Intermediate Points Along the Length of a Bar.—The problems previously treated herein concern axial loads applied at the ends of a bar. Bars with axial loads applied at interior points are considered in Figs. 19 and 20. In Fig. 19, the left part of the bar is in compression



$P = 0.10\ EI/\lambda^2$
$P_{cr} = 9.87\ EI/L^2 = 0.1542\ EI/\lambda^2$

(a)

										Common Factors
w_i from Fig. 16(a)	0	525	840	975	960	825	600	315	0	$\frac{M\lambda^2}{240EI}$

(b)

										Common Factors
Assumed Deflection, w_a from Eq. 4	0	969	1,550	1,799	1,771	1,522	1,107	581	0	$\frac{M\lambda^2}{240EI}$
$w_0 = w_a + w_i$	0	1,494	2,390	2,774	2,731	2,347	1,707	896	0	$\frac{M\lambda^2}{240EI}$
Moments due to P	0	149.4	239.0	277.4	273.1	234.7	170.7	89.6	0	$\frac{M}{240}$
Distributed Angle Changes	0	−149.4	−239.0	−277.4	−273.1	−234.7	−170.7	−89.6	0	$\frac{M}{240EI}$
Assumed Average Slope, Part (1)	750.0	600.6	361.6	84.2	−188.9	−423.6	−594.3	−683.9		$\frac{M\lambda}{240EI}$
Trial Deflection, Part (1)	0	750.0	1,350.6	1,712.2	1,796.4	1,607.5	1,183.9	589.6	−94.3	$\frac{M\lambda^2}{240EI}$
Deflection, Part (2)	0	−12.5	−19.9	−23.1	−22.8	−19.6	−14.2	−7.5	0	$\frac{M\lambda^2}{240EI}$
Linear Correction to Deflection	0	11.8	23.6	35.4	47.2	58.9	70.7	82.5	94.3	$\frac{M\lambda^2}{240EI}$
Resultant Deflection, w'_a	0	749.3	1,354.3	1,724.5	1,820.8	1,646.8	1,240.4	664.6	0	$\frac{M\lambda^2}{240EI}$

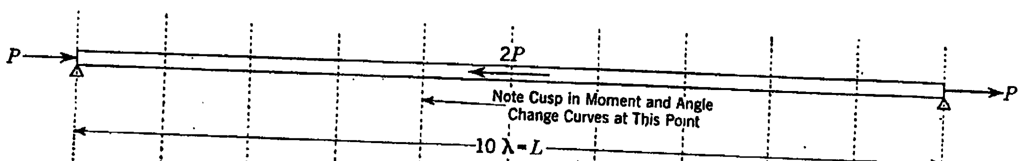
(c) Final Deflections w_a which Lead to Same Deflections w'_a

									Common Factors
0	720	1,319	1,705	1,826	1,672	1,271	685	0	$\frac{M\lambda^2}{240EI}$

(d) Moments in Beam

										Common Factors
Due to M Alone	240	210	180	150	120	90	60	30	0	$\frac{M}{240}$
Due to Thrust	0	124.5	215.9	268.0	278.6	249.7	187.1	100.0	0	$\frac{M}{240}$
Total Moment	240.0	334.5	395.9	418.0	398.6	339.7	247.1	130.0	0	$\frac{M}{240}$
Total Moment	1.000	1.394	1.650	1.742	1.661	1.415	1.030	0.542	0	M

FIG. 18.—DEFLECTION OF A BAR SUBJECTED TO END MOMENT AND DIRECT COMPRESSION



(a) Assumed Parabolic Deflection Curve

																						Common Factors
Assumed Deflection, w_a	0		375		667		875		1,000		1,042		1,000		875		667		375		0	a
Moment	0		175		267		275		200		−42		−200		−275		−267		−175		0	$P a$
Angle Changes	0		−175		−267		−275		−200		+42		+200		+275		+267		+175		0	$P a/E I$
Equivalent Concentrated Angle Changes	0		−168		−260		−268		−172		35		193		268		260		168		0	$P a \lambda/E I$
Assumed Average Slope		404		236		−24		−292		−464		−429		−236		32		292		460		$P a \lambda/E I$
Trial Deflection	0		404		640		616		324		−140		−569		−805		−773		−481		−21	$P a \lambda^2/E I$
Linear Correction to Deflection	0		2		4		6		8		11		13		15		17		19		+21	$P a \lambda^2/E I$
Resultant Deflection, w'_a	0		406		644		622		332		−129		−556		−790		−756		−462		0	$P a \lambda^2/E I$

(b) Final Deflection Curve Leading to a Negative Value of P_{cr}

w_a	0	226	460	710	1,000	1,323	1,547	1,544	1,255	698	0	a
Moment	0	26	60	110	200	−323	−747	−944	−855	−498	0	$P a$
w'_a	0	732	1,491	2,312	3,245	4,271	4,981	4,964	4,027	2,256	0	$-P a \lambda^2/E I$
Ratio $\frac{w_a}{w'_a}$		0.309	0.308	0.307	0.308	0.310	0.311	0.311	0.312	0.309		$-E I/P \lambda^2$

$$P_{cr} = -0.310 \frac{E I}{\lambda^2} = -31.0 \frac{E I}{L^2}$$

(c) Final Deflection Curve Leading to a Positive Value of P_{cr}

w_a	0	480	839	1,009	1,000	899	749	570	388	197	0	a
Moment	0	280	439	409	200	101	51	30	12	3	0	$P a$
w'_a	0	1,078	1,885	2,270	2,260	2,027	1,688	1,296	874	439	0	$P a \lambda^2/E I$
Ratio $\frac{w_a}{w'_a}$		0.445	0.445	0.444	0.442	0.444	0.444	0.440	0.444	0.449		$E I/P \lambda^2$

$$P_{cr} = 0.444 \frac{E I}{\lambda^2} = 44.4 \frac{E I}{L^2}$$

(d) Deflections and Moments for Fig. 19(b) (e) Deflections and Moments for Fig. 19(c)

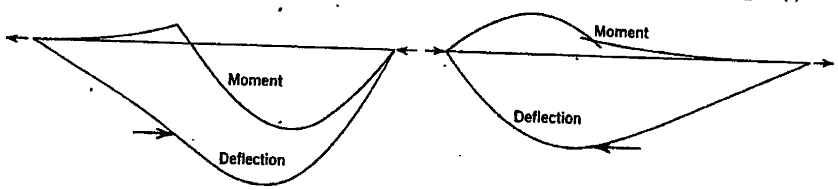
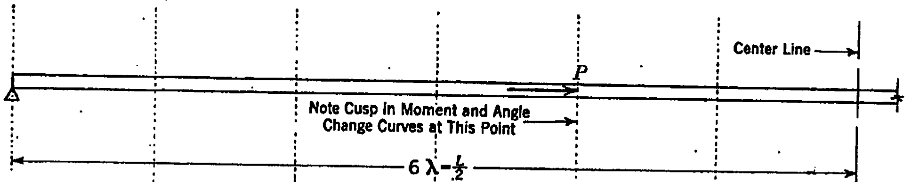


FIG. 19.—BUCKLING OF A BAR SUBJECTED TO AXIAL TENSION AND COMPRESSION



(a) Symmetrical Configuration

														Common Factors
Assumed Deflection, w_a	0		..		..		..		c		$c+1,414$		$c+2,000$	a
Moment	0		0		0		0		0		1,414		2,000	$P a$
Angle Changes	0		0		0		0		0		−1,414		−2,000	$P a/E I$
Equivalent Concentrated Angle Changes	0		0		0		0		−270		−1,345		−1,902	$P a \lambda/E I$
Average Slope		2,566		2,566		2,566		2,566		2,296		951		$P a \lambda/E I$
Resultant Deflection, w'_a									c'		$c'+2,296$		$c'+3,247$	$P a \lambda^2/E I$
Ratio, $\frac{w_a}{w'_a}$, if $\frac{c}{c'}$ has same value as ratio:											0.6159		0.6160	$E I/P \lambda^2$

$$P_{cr} = 0.6159 \frac{E I}{\lambda^2}$$

Deflection, w'_a	0	2,566	5,132	7,698	10,264	12,560	13,511	$P a \lambda^2/E I$

(b) Antisymmetrical Configuration

Assumed Deflection, w_a	0		...		...		...		3,948		2,424		0	a
Moment	0		658		1,316		1,974		2,632		1,766		0	$P a$
Angle Changes	0		−658		−1,316		−1,974		−2,632		−1,766		0	$P a/E I$
Equivalent Concentrated Angle Changes	0		−658		−1,316		−1,974		−2,415.5		−1,691		0	$P a \lambda/E I$
Assumed Average Slope		3,500		2,842		1,526		−448		−2,863.5		−4,554.5		$P a \lambda/E I$
Trial Deflection	0		3,500		6,342		7,868		7,420		4,556.5		2.0	$P a \lambda^2/E I$
Linear Correction to Deflection	0		−0.3		−0.7		−1.0		−1.3		−1.7		−2.0	$P a \lambda^2/E I$
Resultant Deflection, w'_a	0		3,500		6,341		7,867		7,419		4,555		0	$P a \lambda^2/E I$
Ratio, $\frac{w_a}{w'_a}$									0.5321		0.5322			$E I/P \lambda^2$

$$P_{cr} = 0.5322 \frac{E I}{\lambda^2}$$

FIG. 20.—BUCKLING OF A BAR WITH COMPRESSION IN MIDDLE THIRD OF LENGTH

and the right part in tension. The point of application of the interior load is assumed to deflect with the bar; consequently if the load point deflects shears must be applied at the ends of the bar for equilibrium. Since there will be a cusp, or discontinuity in slope of the angle-change diagram at the point of application of the interior load, the procedure used is to write the equivalent concentrated angle changes instead of making the correction that can be made for a smooth angle-change curve. In Fig. 19(*a*) a symmetrical parabolic deflection curve is assumed first. One finds a peculiar result: Some of the resulting deflections are negative. If these deflections are taken as a new deflection curve, and the process repeated, eventually one comes to the result shown in Fig. 19(*b*) where, apparently, the critical load is negative; but this merely indicates a situation in which the left part of the bar is in tension and the right part in compression. It is reasonable that the buckling load should be less for this arrangement of loads since a longer part of the bar is thereby subjected to compression. The final result for the original problem is shown in Fig. 19(*c*). It may be obtained by repeated trials, but not by a process in which each new configuration is the result obtained from a previous assumed configuration, unless the starting point is a configuration not containing any appreciable component of the type obtained in Fig. 19(*b*). The shapes of the final deflection curves and the moment diagrams corresponding thereto are shown in Fig. 19(*d*).

A bar subjected to two opposing loads applied at the third points is illustrated by Fig. 20. An exact solution for this problem is available.[18] The problem is given not only to illustrate the procedure for an unusual case, but also to show what can happen when care is not taken to insure that components of deflection corresponding to the lowest critical buckling load are present in the assumed deflection curve. The loads are assumed to be applied on the axis of the bar even when the bar deflects.

A symmetrical deflection of the bar is shown in Fig. 20(*a*). The deflections outside of the region subjected to compression are immaterial in a consideration of the critical buckling load. It will be noted that the critical load is the same as in Fig. 14(*c*); but some care is necessary in obtaining the proper value of c, the unknown constant part of all the deflections in the region considered. For the final deflection curve c can be obtained easily by taking the complete deflection curve for w'_a and repeating the calculations; but for intermediate steps, c can be chosen as having any value, which complicates the problem of placing a limit on the critical load. Obviously there should be no distortion in the region outside of the central part of the bar, however, and therefore one can always make a fair estimate of the situation in this case.

In Fig. 20(*b*) an antisymmetrical deflection is assumed, and the corresponding critical load is calculated. Here again, the deflections outside of the region subject to compression do not enter into the finding of the critical load. It is of interest and importance that the critical load corresponding to the antisymmetrical deflection is lower than that corresponding to the symmetrical configuration for the arrangement of loads chosen. The bar would actually tend to buckle by more or less of a rotation of the central section. However,

[18] "Über die Knickung eines Balkens durch Längskräfte," by O. Blumenthal, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 17, 1937, pp. 232–244, especially pp. 234–239.

this would not have been discovered if only symmetrical deflection curves had been assumed.

Part IV.—Concluding Remarks

Treatment of Large Deflections.—In all of the problems discussed herein the fundamental relation between deformation and moment has been implicitly assumed to be of the following type:

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} \qquad (7)$$

in which y is the deflection, positive downward. Where deflections are large, Eq. 7 is only approximately correct. One should replace it by the relation:

$$\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = -\frac{M}{EI} \qquad (8a)$$

or

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad (8b)$$

The use of the exact relation offers no serious difficulties. In computing deflections from it one must assume the values of the deflections first, and determine the "angle changes" from modified values of $-\frac{M}{EI}$ (by multiplying by a function of the slopes at various points along the bar). One computes the deflections by a series of successive approximations, in which each step is similar to the various procedures outlined in the paper. However, it is not often necessary to consider such refinements.

Further Applications.—The procedure described herein is applicable to many other problems, since it permits a relatively simple and accurate numerical integration of a class of differential equations.

For example, the problem of a beam on elastic supports can be solved by first assuming a set of deflections, then determining the forces acting on the beam, with the consequent moments and angle changes. From the angle changes, the deflections can be computed. If these are the same as the assumed deflections, the problem is solved. If they are different, the process must be repeated.

The general procedure may also be modified so as to solve the problem of determining the natural period of vibration of a beam, or the critical speed of a shaft. Problems of this kind have been solved previously by similar procedures.[5,6] The use of the present modification is to produce a more accurate solution with generally less effort.

Conclusion.—The numerical procedure described herein permits a simple and rapid calculation of deflections of beams and columns and of critical buckling loads for columns with a high degree of accuracy. The method can be extended to other problems of the same mathematical nature.

Acknowledgment

The method of analysis described herein was developed as part of the writer's work in the Engineering Experiment Station of the University of

Illinois, Urbana, Ill. Particular acknowledgment is due Harold Crate, Research Graduate Assistant in Civil Engineering, for assistance in making and checking the calculations, and for extended studies of the procedure.

APPENDIX

Derivation of Formulas for Equivalent Concentrations

In Fig. 5(a) the origin of coordinates is at a, with x positive to the right, and q represents the magnitude of the load at any point, positive upward. Let $z = \frac{x}{\lambda}$, in order to obtain a dimensionless coordinate, and consider the curve of loading with ordinates a, b, c at $z = 0, 1, 2$, respectively, to be a second-degree function of z or of x.

It can be readily verified that Eq. 9 represents a general second-degree function of z having the required values $q = a, b, c$ at $z = 0, 1, 2$, respectively:

$$q = \tfrac{1}{2} a (z - 1)(z - 2) - b z (z - 2) + \tfrac{1}{2} c z (z - 1) \quad \ldots\ldots (9)$$

Then, from statics the equivalent concentrated loads R_{ab} and R_{ba} are determined by the equations:

$$R_{ab} + R_{ba} = \lambda \int_0^1 q\, dz \quad \ldots\ldots (10a)$$

and

$$R_{ba} = \lambda \int_0^1 z\, q\, dz \quad \ldots\ldots (10b)$$

Evaluation of the integrals yields the results

$$R_{ba} = \frac{\lambda}{24}(3a + 10b - c) \quad \ldots\ldots (11a)$$

and

$$R_{ab} = \frac{\lambda}{24}(7a + 6b - c) \quad \ldots\ldots (11b)$$

By analogy, one finds the value of R_{bc}:

$$R_{bc} = \frac{\lambda}{24}(3c + 10b - a) \quad \ldots\ldots (12)$$

from which is readily obtained the value of R_b:

$$R_b = R_{ba} + R_{bc} = \frac{\lambda}{12}(a + 10b + c) \quad \ldots\ldots (13)$$

Similar formulas can be written for a loading curve of higher degree in x or z in terms of ordinates at more points. One may also derive expressions for R in terms of differences or of central differences by expressing q in such terms. Furthermore, one may develop corresponding equations when the segments into which the loading curve is divided are not of equal length. However, for practical purposes Eqs. 11 and 13 are all that are generally needed.

DISCUSSION

Bruce Johnston,[19] Assoc. M. Am. Soc. C. E.—The numerical procedure presented by Professor Newmark has advantages of simplicity, accuracy, and speed that make application to actual design work particularly effective. In an extension course given at Lehigh University, in Bethlehem, Pa., the writer has had the opportunity of presenting the method in detail to a number of engineers. Several of these engineers have found the procedure superior to other similar methods. The procedure was recently applied in connection with the analysis and design of several unusual mill building frames that are now (May, 1942) under construction.

As stated by the author (see "Synopsis"), "The essential features of the procedure are not new"—they are based on the well-known relations between the geometry of a bent beam and its moment-stiffness ratio. The importance of the procedure is not its newness, but its usability in actual design. It reduces the analysis of bending and buckling of struts to a systematic and accurate procedure of arithmetic, with a minimum chance of computational errors, and is exact enough for most applications. In actual structural members the moment of inertia frequently varies in a manner that makes actual integration of the fundamental differential equations exceedingly complex, if not impossible. Simple numerical procedures such as the author's deserve relatively more attention in structural engineering literature than they now have.

The practical usefulness of the procedure in continuous frame analysis will be increased if a summary is made of its relation to the slope-deflection and moment-distribution procedures for obtaining terminal moments of members in framed structures. In Fig. 21 is shown a rotation notation for the angle changes due to unit positive moments applied at either end of a simply supported member. Moments are assumed as positive when they apply a clockwise couple to the end of the beam. The angles of rotation of the end tangents of the beam axis are also considered positive when clockwise. The first subscript indicates the location of the angle change and the second subscript indicates the location of the applied unit moment—that is, ϕ_{AB} = angle change at A due to the unit moment at B.

By the law of reciprocal deflections, $\phi_{AB} = \phi_{BA}$. The three independent angle changes ϕ_{AA}, ϕ_{BB}, and ϕ_{AB} may be determined by model analysis or by two applications of the simple numerical procedure described by the author and illustrated in Figs. 10 and 11. In the symmetrical member $\phi_{AA} = \phi_{BB}$, and only one application would be necessary. The angle changes ϕ'_A and ϕ'_B, due to any applied load (also shown in Fig. 21), are determined easily by one additional application of the author's numerical procedure. Note the difference in the sign convention for the terminal moments M_A and M_B; but the sign of the angle changes will be the same as that of the end slopes in the author's paper. These five angle changes determine any or all of the coefficients needed in a generalized solution either by slope deflection or moment

[19] Senior Engr., Johns Hopkins Laboratory of Applied Physics, Silver Spring, Md.

distribution. The effect of direct load upon the bending stiffness could be included, but is usually neglected in bridge and building frame analysis.

The positive rotation notation for moments, shears, angle changes, and lateral translation of the ends of any member in a loaded frame is shown in

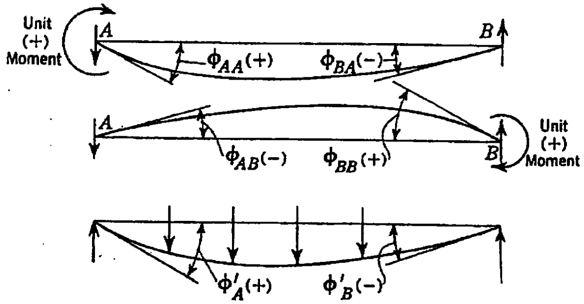


FIG. 21.—END-ANGLE CHANGES DETERMINED BY NUMERICAL PROCEDURE

Fig. 22. In the case of the member with uniform cross section, the "slope-deflection" equations are written:

$$M_{AB} = \frac{2\,E\,I}{l}\left(2\,\theta_A + \theta_B - \frac{3\,\Delta}{l}\right) \pm M_{FA} \quad \text{.........(14a)}$$

and

$$M_{BA} = \frac{2\,E\,I}{l}\left(2\,\theta_B + \theta_A - \frac{3\,\Delta}{l}\right) \pm M_{FB} \quad \text{.........(14b)}$$

in which M_{FA} and M_{FB} are "fixed-end" moments due to loads on the beam span. For downward loads on a horizontal member, M_{FA} is negative and M_{FB} is positive.

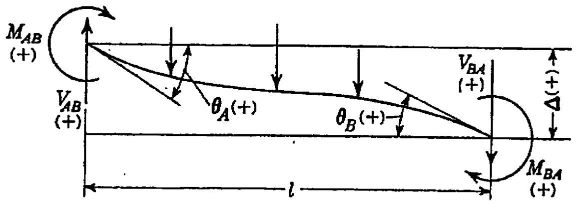
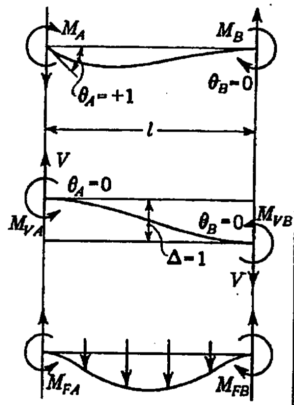
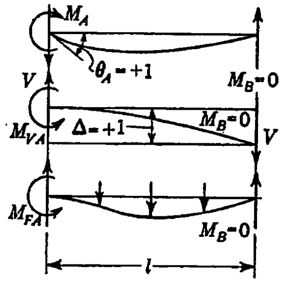
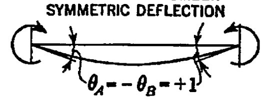
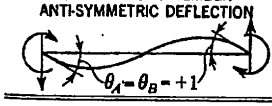


FIG. 22.—MOMENTS, SHEARS, ANGLE CHANGES, AND LATERAL TRANSLATION OF ANY FRAMED MEMBER, SHOWN AS POSITIVE

It may be shown by the "moment-area" relations that the following slope-deflection equations obtain for the general case of variable I, written in terms of the fundamental angle changes shown in Fig. 21:

$$M_{AB} = \frac{1}{\phi_{AA}\,\phi_{BB} - \phi^2_{AB}}\Big[\phi_{BB}\,\theta_A - \phi_{AB}\,\theta_B + (\phi_{AB} - \phi_{BB})\frac{\Delta}{l} + \phi_{AB}\,\phi'_B - \phi_{BB}\,\phi'_A\Big] \quad \text{..........(15a)}$$

TABLE 1.—MOMENT DISTRIBUTION FACTORS FOR END A OF ANY MEMBER AB

				UNIFORM SECTION
STANDARD CASE-FAR END HELD FIXED 	CARRY-OVER FACTOR	$r_{AB}=\left\|\frac{M_B}{M_A}\right\| \begin{matrix}\theta_A=+1\\ \theta_B=0\\ \Delta=0\end{matrix}$	$r_{AB}=-\frac{\phi_{AB}}{\phi_{BB}}$	$\frac{1}{2}$
	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ \theta_B=0\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1}{\phi_{AA}+r_{AB}\phi_{AB}}$	$\frac{4EI}{l}$
	SHEAR STIFFNESS	$S_{VAB}=\left\|V\right\| \begin{matrix}\theta_A=0\\ \theta_B=0\\ \Delta=+1\end{matrix}$	$S_{VAB}=\frac{1+2r_{AB}+\frac{r_{AB}}{r_{BA}}}{l^2(\phi_{AA}+r_{AB}\phi_{AB})}$	$\frac{12EI}{l^3}$
	MOMENT DUE TO UNIT SIDESWAY	$M_{VA}=\left\|M_A\right\| \begin{matrix}\theta_A=0\\ \theta_B=0\\ \Delta=+1\end{matrix}$	$M_{VA}=\frac{-(1+r_{AB})}{l(\phi_{AA}+r_{AB}\phi_{AB})}$	$\frac{6EI}{l^2}$
Note-Factors at End B May Be Obtained by Interchanging Subscripts A and B	FIXED-END MOMENT	$M_{FA}=\left\|M_A\right\| \begin{matrix}\theta_A=0\\ \theta_B=0\\ \Delta=0\end{matrix}$	$M_{FA}=\frac{-(r_{AB}\phi'_B+\phi'_A)}{\phi_{AA}+r_{AB}\phi_{AB}}$	$-\frac{4EI}{l}\times\left(\phi'_A+\frac{\phi'_B}{2}\right)$
SPECIAL CASE, FAR END SIMPLY SUPPORTED 	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ M_B=0\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1}{\phi_{AA}}$	$\frac{3EI}{l}$
	SHEAR STIFFNESS	$S_{VAB}=\left\|\frac{-M_A}{l}\right\| \begin{matrix}\theta_A=0\\ M_B=0\\ \Delta=+1\end{matrix}$	$S_{VAB}=\frac{1}{\phi_{AA}l^2}$	$\frac{3EI}{l^3}$
	FIXED-END MOMENT	$M_{FA}=\left\|M_A\right\| \begin{matrix}\theta_A=0\\ M_B=0\\ \Delta=0\end{matrix}$	$M_{FA}=\frac{-\phi'_A}{\phi_{AA}}$	$-\frac{3EI\phi'}{l}$
SYMMETRICAL MEMBER SYMMETRIC DEFLECTION 	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ \theta_B=-1\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1-r_{AB}}{\phi_{AA}+r_{AB}\phi_{AB}}$ $S_{MAB}=S_{MBA}$ (Symmetrical Member)	$\frac{2EI}{l}$
SYMMETRICAL MEMBER ANTI-SYMMETRIC DEFLECTION 	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ \theta_B=+1\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1+r_{AB}}{\phi_{AA}+r_{AB}\phi_{AB}}$ $S_{MAB}=S_{MBA}$ (Symmetrical Member)	$\frac{6EI}{l}$

and

$$M_{BA} = \frac{1}{\phi_{AA}\,\phi_{BB} - \phi^2_{AB}} \left[\phi_{AA}\,\theta_B - \phi_{AB}\,\theta_A + (\phi_{AB} - \phi_{AA})\frac{\Delta}{l} + \phi_{AB}\,\phi'_A - \phi_{AA}\,\phi'_B \right] \qquad (15b)$$

The factors commonly used in the moment-distribution procedure, as defined by the general slope-deflection equations (Eq. 15), are given in Table 1.

In the actual calculation of the basic end-angle changes ϕ_{AA}, ϕ_{BB}, ϕ_{AB}, ϕ'_A, and ϕ'_B, the procedure used by the author in Figs. 10 and 11 may be simplified. Deflections need not be calculated at all. The end slopes are simply equal to the "conjugate" beam-end reactions caused by the "equivalent concentrated angle changes" treated as loads. Upward end reaction is positive on the left end and negative on the right end. The equivalent concentrated angle-change loads should be calculated by the formulas given in Figs. 3 and 5, and the alternate to the author's procedure in Fig. 10 is presented in Fig. 23. The results check those of Fig. 10. If a mechanical calculating machine is used in the computation, it would be convenient to divide the member into either five or ten segments. The calculation of end reactions is then obtained by multiplying the equivalent concentrated angle-change loads by successive decimal fractions, 0.2, 0.4, etc., or 0.1, 0.2, 0.3, etc., up to 1.0, in the case of five or ten segments, respectively.

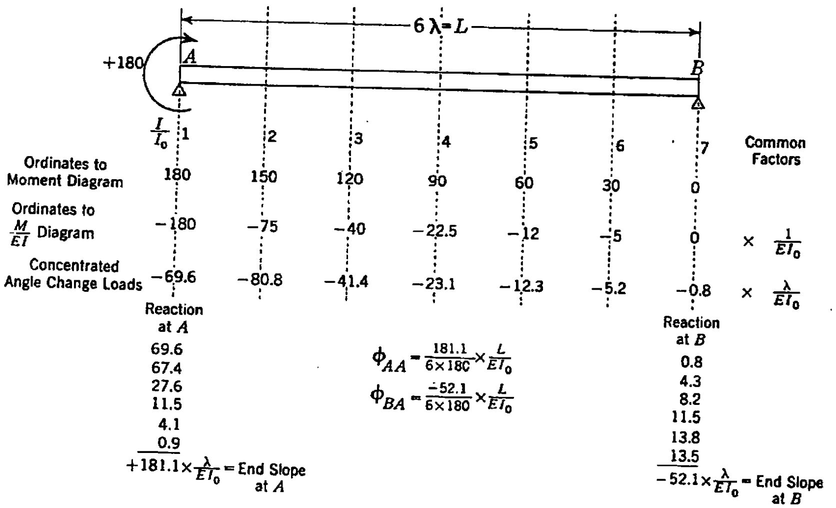


FIG. 23.—ALTERNATE PROCEDURE TO FIG. 10 FOR CALCULATION OF END SLOPES ONLY

The other angle changes ϕ_{BB}, ϕ'_A, and ϕ'_B (also a check on ϕ_{AB}) are obtained by two additional sets of computation similar to Fig. 23. It should be noted that in computing ϕ_{BB} and ϕ_{AB} ($= \phi_{BA}$) the signs of the results and the sense of the applied moment at the right end of the beam will be reversed from that shown in Fig. 11 to conform to the rotation sign convention used in this discussion. After the basic angle changes are obtained, they may be substitued in Eq. 15 or in Table 1 to provide the necessary basis for analysis either by the slope-deflection or moment-distribution procedures, respectively. These details will be obvious to one already familiar with structural frame analysis.

The writer has discussed the application of the paper to structural frame analysis in cases where direct stress in a member may be neglected in so far as its effect on bending is concerned. The author's procedure is particularly adapted to the computation of critical buckling loads under direct stress for cases of nonuniform cross section. The method is clearly outlined by the author. The procedure furnishes the engineer with a simple method involving only the processes of arithmetic, and thereby bears a relation to the elastic stability theory of bars similar to that which the Hardy Cross moment-distribution method bears to older and more cumbersome methods of structural frame analysis.

M. S. KETCHUM, JR.,[20] ASSOC. M. AM. SOC. C. E.—The author's paper is another contribution to a distinctly American tradition of structural analysis founded on the work of Hardy Cross. This tradition is characterized by a nonmathematical solution of quite difficult cases in such a manner that the structural behavior is always evident in quantitative terms.

The writer would like to suggest a simple and convenient approximation that he has found to be useful in a large variety of buckling problems, including the lateral buckling of crane runway girders. The total deflection, w_0, in the strut analyzed in Fig. 17 may be estimated as soon as the deflection, w_i, due to the lateral load, is calculated and before the value of the critical load, P_{cr}, is determined. The assumption may be made that the shape of the bending-moment diagram for the moments, $P\,w_0$, due to the column load, is of the same shape as the bending moments due to lateral loads.

If M_L is represented as the moment due to the lateral load and w_i as the deflection due to this load, then the deflection, w_a, due to a moment, $P\,w_0 = P\,(w_i + w_a)$, is:

$$w_a = \frac{w_i\,P\,(w_i + w_a)}{M_L} \qquad (16)$$

Solving for w_0 in terms of w_i:

$$w_0 = \frac{w_i}{1 - \dfrac{w_i\,P}{M_L}} \qquad (17)$$

A substitution of the values of w_i, P, and M_L at the center of the strut gives:

$$\frac{w_i\,P}{M_L} = \frac{1{,}042 \times 0.02}{100} = 0.2084; \text{ and}$$

$$w_0 = \frac{1{,}042\,q\,\lambda^4}{8\,E\,I}\left(\frac{1}{1 - 0.2084}\right) = \frac{1{,}320\,q\,\lambda^4}{8\,E\,I} \qquad (18)$$

[20] Asst. Prof., Structural Eng., Case School of Applied Science, Cleveland, Ohio.

The corresponding value from Fig. 17 has a coefficient of 1,306 so that the approximation is satisfactory for most problems.

The critical buckling load for this strut determined in Fig. 12 also may be estimated by the condition that $w_i = w_0$, and

$$w_0 = \frac{w_i \, P \, w_0}{M_L} \quad \text{(19a)}$$

and

$$P_{cr} = \frac{M_L}{w_i} \quad \text{(19b)}$$

Substitution of values from Fig. 17 gives:

$$P_{cr} = \frac{100 \, E \, I}{1{,}042 \, \lambda^3} = \frac{9.60 \, E \, I}{L^2} \quad \text{(20)}$$

The corresponding coefficient from Fig. 12 is 9.87 so that the approximation is in fair agreement.

The author is to be commended for his simple yet thorough and rigorous treatment of deflection and buckling. This paper should dispel much of the mystery that surrounds all but the most simple column problems.

JOHN B. WILBUR,[21] ASSOC. M. AM. SOC. C. E.—Professor Newmark's paper has been stimulating to the writer, particularly since it places emphasis on mathematically simple approaches to problems in elastic stability. Although Professor Newmark shows that simultaneous equations may be used to insure that the assumed deflection curve of a compression strut is essentially proportional to its shape as computed elastically on the basis of the assumed deflection curve, his detailed treatment is devoted to the substitution of a method of successive approximations for the solution of simultaneous equations.

The writer solved the problem illustrated by Fig. 13 on the basis of five simultaneous equations, and obtained substantially the same results. This solution was relatively straightforward, although the form of the simultaneous equations was such that it was convenient to adopt a method of successive approximations in their solution. Having successfully completed this solution, it was then decided to investigate the accuracy of simpler direct solutions, based on two simultaneous equations only. Symmetrical cases only were considered; otherwise three equations would have been necessary. The results of these relatively simple solutions were quite satisfactory, as is illustrated in the following discussion.

Consider first the case of Fig. 13. Assuming that the deflections of the strut at the center and quarter points are δ and $a\,\delta$, respectively, the equivalent concentrated $\frac{M}{E\,I}$-loads at the center and quarter points are computed from the relations for parabolic loading given in Fig. 5(b). These loads, together with the resultant net reaction, are shown in Fig. 24. By the moment area

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method:

$$a\,\delta = \frac{P\,L\,\delta}{48\,E\,I}(11\,a + 6)\frac{L}{4} = \frac{P\,L^2\,\delta}{192\,E\,I}(11\,a + 6) \quad \text{(21a)}$$

and

$$\delta = \frac{P\,L\,\delta}{48\,E\,I}(11\,a + 6)\frac{2\,L}{4} - \frac{P\,L\,\delta}{48\,E\,I}(10\,a + 1)\frac{L}{4} = \frac{P\,L^2\,\delta}{192\,E\,I}(12\,a + 11) \quad \text{(21b)}$$

Let $A = \frac{P\,L^2}{192\,E\,I}$ and Eqs. 21 become

$$a = A\,(11\,a + 6) \quad \text{(22a)}$$

and

$$1 = A\,(12\,a + 11) \quad \text{(22b)}$$

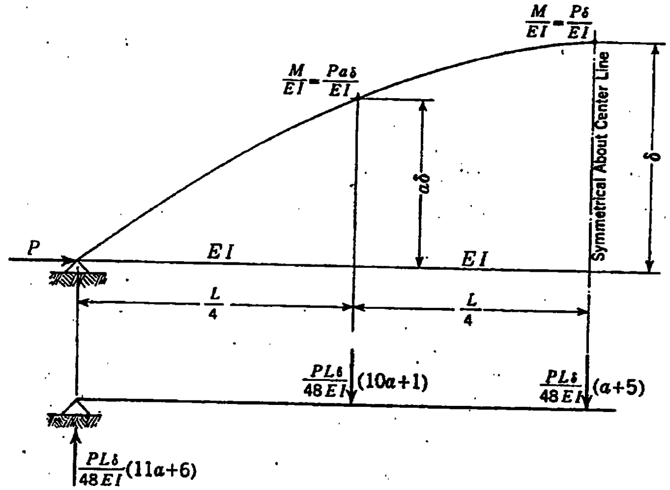


FIG. 24

Solving simultaneously: $a = 0.707$; $A = \frac{1}{19.48} = \frac{P\,L^2}{192\,E\,I}$. Hence, $P_{cr} = \frac{192}{19.48}\frac{E\,I}{L^2} = \frac{9.86\,E\,I}{L^2}$ vs. $\frac{\pi^2\,E\,I}{L^2}$ (exactly).

Before applying this procedure to the case illustrated by Professor Newmark in Fig. 15, where I is not constant throughout the length of the strut, it was convenient to develop equations for equivalent concentrated loads for parabolic loading curves for the case in which the spans of the two adjacent segments are not equal. Referring to Fig. 25, these equations are as follows:

$$R_{bc} = \frac{l_2}{12\,l_1\,(l_1 + l_2)}\left[a\,(-\,l_2^2) + b\,(4\,l_1 + l_2)(l_1 + l_2) + c\,l_1\,(2\,l_1 + l_2)\right] \quad \text{(23a)}$$

and

$$R_{ba} = \frac{l_1}{12\,l_2\,(l_1 + l_2)}\left[a\,l_2\,(l_1 + 2\,l_2) + b\,(l_1 + 4\,l_2)(l_1 + l_2) + c\,(-\,l_1^2)\right] \quad \text{(23b)}$$

Assuming that the deflections of the strut at the center and the 0.2-points are δ and $a\,\delta$, respectively, the equivalent concentrated $\frac{M}{E\,I}$-load at the 0.2-point is computed from the foregoing relations for parabolic loadings, whereas

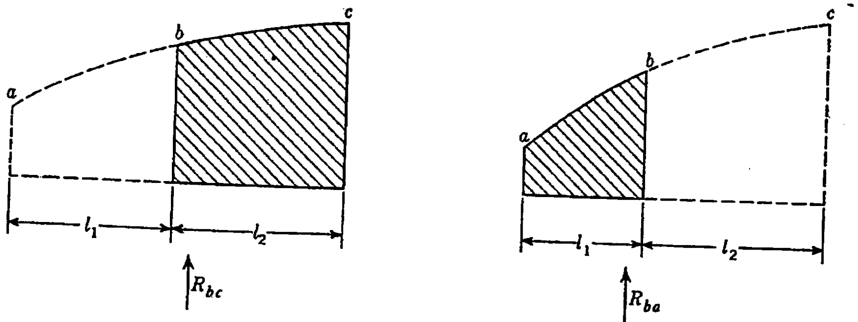


FIG. 25

the equivalent concentrated $\frac{M}{E\,I}$-load at the center is computed on the basis of parabolic loadings of equal adjacent sections. These loads, together with the resultant net reaction, are shown in Fig. 26. By the moment-area method,

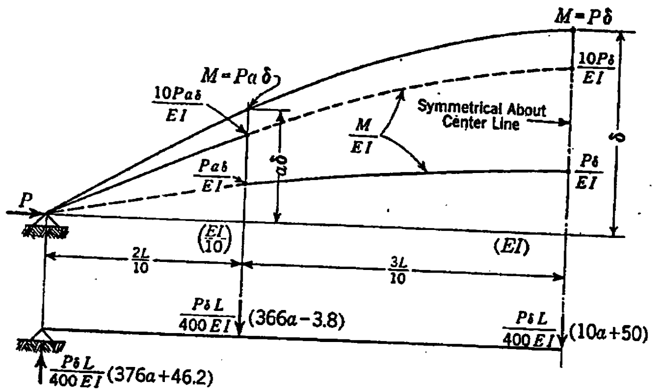


FIG. 26

$$a\,\delta = \frac{P\,\delta\,L}{400\,E\,I}(376\,a + 46.2)\frac{2\,L}{10} = \frac{P\,L^2\,\delta}{4{,}000\,E\,I}(752\,a + 92.4)\ldots(24a)$$

and

$$\delta = \frac{P\,\delta\,L}{400\,E\,I}(376\,a + 46.2)\frac{5\,L}{10} - \frac{P\,\delta\,L}{400\,E\,I}(366\,a - 3.8)\frac{3\,L}{10}$$

$$= \frac{P\,L^2\,\delta}{4{,}000\,E\,I}(782\,a + 219.6)\ldots\ldots\ldots\ldots(24b)$$

Let $A = \frac{P\,L^2}{4{,}000\,E\,I}$, and Eqs. 24 become

$$a = A\,(752\,a + 92.4)\ldots\ldots\ldots\ldots(25a)$$

and

$$1 = A\,(782\,a + 219.6)\ldots\ldots\ldots\ldots(25b)$$

Solving simultaneously: $a = 0.823$; $A = \frac{1}{863} = \frac{P\,L^2}{4{,}000\,E\,I}$. Hence, $P_{cr} = \frac{4{,}000}{863}\frac{L^2}{E\,I} = 4.64\frac{L^2}{E\,I}$ vs. $\frac{4.5\,L^2}{E\,I}$ (exactly).

Since, for a strut composed of sections of constant $E\,I$, the deflection curve must be compounded of sine curves, an interesting exact solution to the foregoing problem may be made as follows, with reference to Fig. 27.

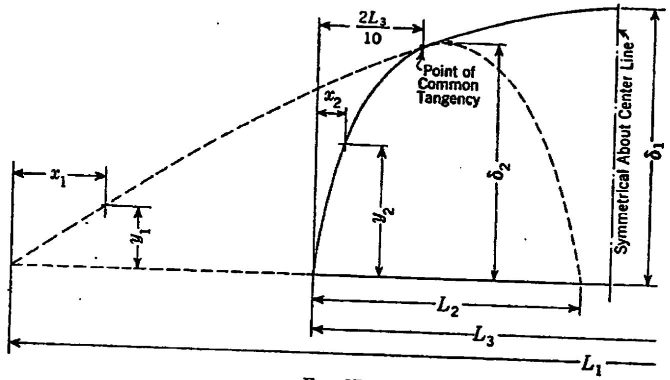


FIG. 27

For the case under consideration, the true deflection curve, shown by the full line, is composed of segments of the curves

$$y_1 = \delta_1 \sin\frac{\pi\,x_1}{L_1}\ldots\ldots\ldots\ldots(26a)$$

and

$$y_2 = \delta_2 \sin\frac{\pi\,x_2}{L_2}\ldots\ldots\ldots\ldots(26b)$$

At the point of common tangency, where

$$x_1 = \frac{L_1 - L_3}{2} + \frac{2\,L_3}{10} = \frac{1}{10}(5\,L_1 - 3\,L_3)\ldots\ldots\ldots(27a)$$

and

$$x_2 = \frac{L_3}{5}\ldots\ldots\ldots\ldots(27b)$$

the following relations hold:

$$y_1 = y_2;\ \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2};\ \text{and}\ \frac{d^2y_1}{dx_1^2} = \frac{1}{10}\frac{d^2y_2}{dx_2^2},\ \text{since}\ I_1 = 10\,I_2.$$

These relations reduce to

$$\delta_1 \sin \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \delta_2 \sin \frac{\pi}{5} \frac{L_3}{L_2} \qquad (28a)$$

$$\frac{\delta_1}{L_1} \cos \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \frac{\delta_2}{L_2} \cos \frac{\pi}{5} \frac{L_3}{L_2} \qquad (28b)$$

and

$$\frac{\delta_1}{L^2_1} \sin \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \frac{\delta_2}{10\,L^2_2} \sin \frac{\pi}{5} \frac{L_3}{L_2} \qquad (28c)$$

Dividing Eq. 28c by Eq. 28a, $\frac{1}{L^2_1} = \frac{1}{10\,L^2_2}$; whence $L_2 = \frac{L_1}{\sqrt{10}}$. Eqs. 28a and 28b become

$$\delta_1 \sin \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \delta_2 \sin \frac{\pi\,\sqrt{10}}{5} \frac{L_3}{L_1} \qquad (29a)$$

and

$$\delta_1 \cos \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \delta_2\,\sqrt{10} \cos \frac{\pi\,\sqrt{10}}{5} \frac{L_3}{L_1} \qquad (29b)$$

Dividing Eq. 29a by Eq. 29b,

$$\sqrt{10} \tan \frac{\pi}{10} \frac{(5\,L_1 - 3\,L_3)}{L_1} = \tan \frac{\pi\,\sqrt{10}}{5} \frac{L_3}{L_1} \qquad (30)$$

and $\frac{L_3}{L_1} = 0.675$. Hence, $L_1 = 1.481\,L_3$. This gives the "reduced" length of the strut under consideration. The critical load is then given by

$$P_{cr} = \frac{\pi^2\,E\,I}{(1.481\,L_3)^2} = \frac{4.50\,E\,I}{L^2_3} \qquad (31)$$

In the opinion of the writer, Professor Newmark has performed a definite service since he has dissociated one type of problem in elastic stability from the shroud of the formal solution of differential equations. It is to be hoped that relatively simple methods of solution will eventually be developed for other types of problems in elastic stability.

RALPH W. STEWART,[22] M. AM. SOC. C. E.—Effective procedures for determining the elastic curves of beams are presented in compact form by the author. The paper has value in an engineer's reference library since the analyses demonstrated have heretofore been scattered in different treatises. The use of progressive load, shear, and moment increments to establish deflections and the alinement of elastic curves is familiar in the analysis of arches. An excellent demonstration of the use of the device termed "linear correction to moments" (which, by analogy, is the same as "linear correction to deflections") has been presented by A. W. Buel, M. Am. Soc. C. E., and C. S. Hill.[23]

[22] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

[23] "Reinforced Concrete," by A. W. Buel and C. S. Hill, The Engineering News Publishing Co., New York, N. Y., 1906, Fig. 43, p. 140.

It appears to the writer that a part of the paper dealing with beams of variable section can be improved in analytical procedure and also that a rather important deficiency in the author's illustration of the use of his computations justifies a rewriting of this part of the paper.

To clarify and justify this opinion, a statement regarding the basic constants of beam flexure is necessary. A basic constant is defined as being either a simple constant quantity or a constant ratio between two variable quantities. A constant that is more complex than a basic constant will be referred to as a derived constant because it is derived by the use of two or more basic constants.

The basic constants of beam flexure are well illustrated by the dimensions governing railway curves. A circular railway curve is completely determined by the length of a tangent and the angle of intersection of the tangents. The transition spiral often used at the end of a circular curve is fully determined by the length of its tangents and their angle of intersection. If the over-all length of the spiral curve is known (as the span of a beam is known), then one tangent and the angle of intersection are sufficient. The basic constants governing the flexure of a beam that has no more than two supports are similar and are as follows, the distance between supports being known:

(1) The basic stiffness of the left end of the beam. This is the ratio of the moment to the angle of intersection between the tangents to the elastic curve when the beam is hinged at the right end and a moment is applied at the left end.

(2) The ratio of the length of one tangent to the elastic curve to the length of the beam for the same condition of flexure as in (1).

(3) and (4) These are the same as (1) and (2), except that the left end of the beam is taken as hinged and the moment is applied at the right end.

(5) The angle of intersection between the tangents of the elastic curve when the beam acts as a simply supported beam subjected to its loads.

(6) The ratio of the length of one tangent of the elastic curve to the length of the beam for the simply supported condition.

The first four constants are "beam constants," which are independent of the loading. The last two are "load constants," which depend on the loads.

Of the six basic constants of flexure, only five are independent, since the principle of Maxwell's theorem of reciprocal deflections (angular) will enable any missing beam constant to be computed from the others. It is understood in beam flexure that the tangents may be taken as equal to their projected length in the unsprung beam.

An end slope is not a basic flexure constant. This can be seen from the fact that in a cantilever beam the slope at the fixed end is known to be zero; but this knowledge, combined with the length of a tangent to the elastic curve, is insufficient to determine the curve. If, however, the length of a tangent and the intersection angle between tangents are known, the curve is determined.

The Hardy Cross stiffness factor is not a basic constant because it is a function of an end slope and also two basic stiffnesses, one at each end of the beam.

Much valuable time has been lost and is still being lost by engineering designers through injudicious selection of flexure constants in analyzing the various types of structures. The author's treatment of the beam in Figs. 10 and 11 would tend to encourage rather than correct this tendency. A revised

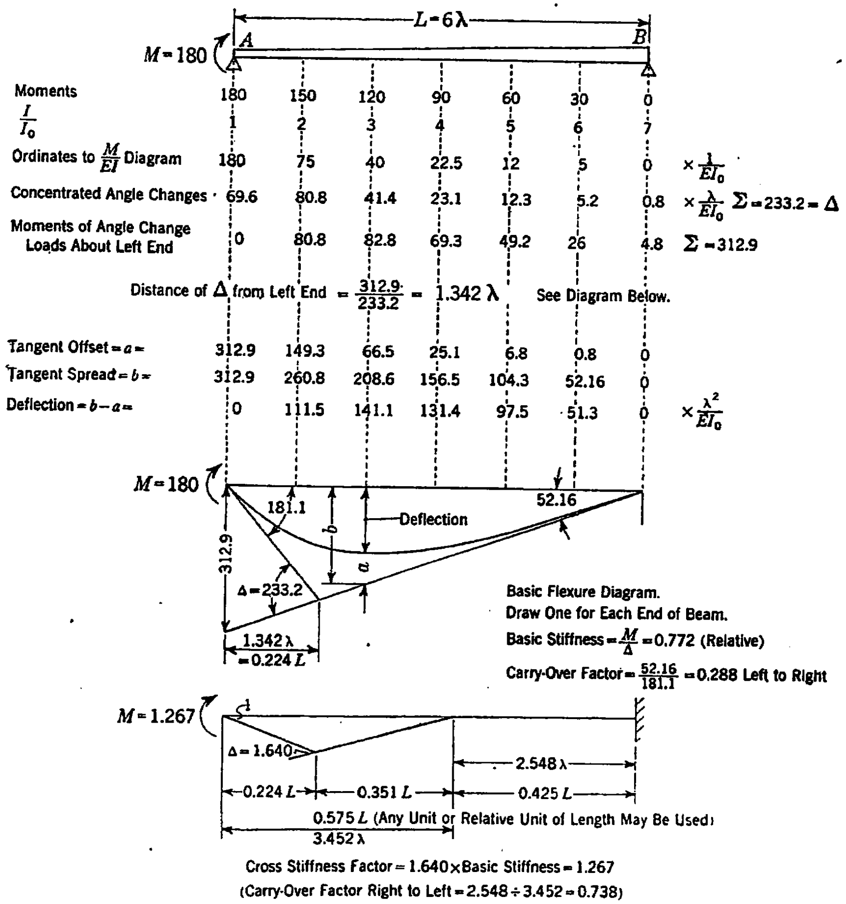


FIG. 28

computation of the properties of this beam is therefore offered in Fig. 28, and the justification for this revised computation is illustrated by Fig. 29. The columns in Fig. 29 are the same members whose properties are determined in Fig. 28. The deck is of constant section with relative stiffness as shown. The sidesway moments and the appurtenant lateral force as shown in Fig. 29 are computed by the following consecutive steps:

Write in deck traverse angles 1, 2, and 3. Multiply stiffness, 2.97, by angle value, 3, to get moment 8.91; divide moment 8.91 by stiffness 1.485 to

get column traverse angle 6. Add angles 1 and 6 to get bottom column traverse angle 7; multiply angle 7 by stiffness 0.772 to get moment 5.404. Add all column moments and divide by frame height to get lateral force. The effect of any other lateral force will be in direct proportion. The lateral deflection of the structure follows as a direct by-product.

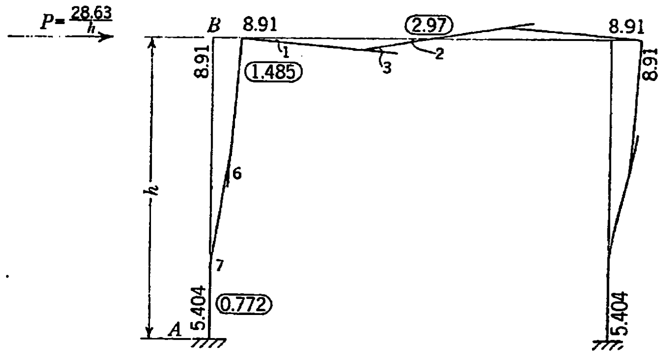


FIG. 29

It can be seen that the foregoing computation of sidesway moments, using basic constants, can be done in less than one minute, whereas computing these moments, with the derived constants, which are the only ones mentioned by the author, would take much longer, would give a less accurate result, and would fail to give the deflection of the structure as a direct by-product. The same situation would prevail with vertical loads.

The usefulness of the computation of deflections and elastic properties of an individual beam is obviously increased by bringing the results into a form that can be used for the quick computation of the deflections and moments of a structure in which the beam becomes a member.

STEFAN J. FRAENKEL,[24] JUN. AM. SOC. C. E.—Recently the writer was called upon to investigate the stresses in a 125-ft derrick boom. Since such long and slender members are subjected to considerable deformation which, in turn, influences the stresses, it was necessary to determine the deflection of this boom. The methods outlined in Professor Newmark's able paper lent themselves well to this investigation.

The boom (see Fig. 30) consisted of four angles—two 6-in. by 4-in. by ½-in. angles, which formed the top chord, and two 4-in. by 4-in. by ½-in. angles, which constituted the bottom chord. A preliminary investigation indicated that the strengthening of the middle section (which was 59 ft long) with 3-in. by ½-in. bars at the top and 3-in. by ⅜-in. bars at the bottom was advisable. That is the section shown in Fig. 30, which was also used in the investigation related herein. The section of the boom varied considerably outside the middle part, as is shown by the variations in the values of I and y.

[24] With Eng. Dept., Pittsburgh-Des Moines Steel Co., Pittsburgh, Pa.

A general view of the boom is shown in Fig. 31. The boom will be analyzed in a horizontal position. Because of the fact that its cross section is unsymmetrical, the neutral axis does not coincide with the geometrical center line,

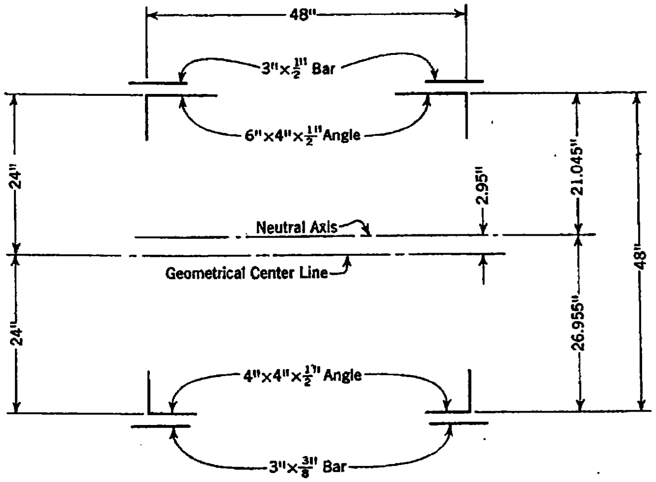


FIG. 30.—SECTION IN CENTER PART OF BOOM

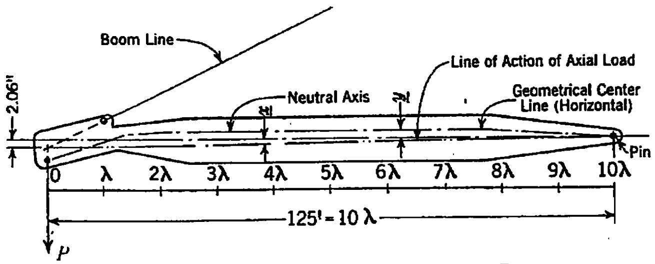


FIG. 31.—GENERAL VIEW OF 125-FT DERRICK BOOM

the distance between them at any point being denoted by y. The line of action of the axial loads is determined at the right end of the boom by the location of the pin, and at the left end by the intersection of the boom line and

TABLE 2.—VALUES OF x, y, AND I FOR VARIOUS VALUES OF λ

Symbol	0	1 λ	2 λ	3 λ	4 λ	5 λ	6 λ	7 λ	8 λ	9 λ	10 λ
x	2.06	1.86	1.65	1.45	1.24	1.03	0.83	0.62	0.41	0.21	0.00
y	−8.39	1.13	2.27	2.95	2.95	2.95	2.95	2.95	2.27	1.76	0.00
I	42,057	5,363	7,098	11,843	11,843	11,843	11,843	11,843	7,098	3,387	5,071
I	12.4 I_o	1.59 I_o	2.09 I_o	3.49 I_o	3.49 I_o	3.49 I_o	3.49 I_o	3.49 I_o	2.09 I_o	1.00 I_o	1.50 I_o

the vertical line of action of the load P. (It should be understood that P includes both live load and the left reaction of the boom due to its own weight.) This intersection (which, of course, is imaginary) occurs for the case of the flat boom 2.06 in. below the geometrical center line. The distance at any point between the geometrical center line and the line of action of the axial load will be denoted by x. A summary of the values of x, y, and I for various values of λ is given in Table 2.

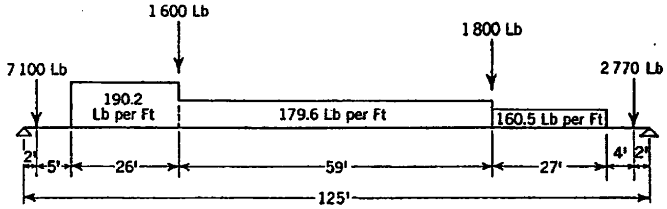


FIG. 32.—LOADING SKETCH OF BOOM

Fig. 32 shows the loading of the boom due to its own weight. The dead-load reaction at the left end causes an axial load of 51,250 lb, and the maximum axial load due to live load is 64,950 lb. In computing the deflections the following conventions were adopted:

(1) Downward deflection is positive;

(2) Inch and pound units are employed in the computations;

(3) The modulus of elasticity, E, is taken as 30 by 10^6 lb per sq in.;

and

(4) The boom is divided into ten parts, so that $10\,\lambda = 125$ ft.

Three deflections are:

(a) δ_w = deflection caused by bending moment due to the weight of the boom;

(b) δ_a = deflection due to axial dead load; and

(c) δ = deflection due to axial live load.

The following relations hold:

$$\delta_w + \delta_a = \Delta_D \qquad (32a)$$

and

$$\delta_w + \delta = \Delta \qquad (32b)$$

in which, Δ_D is the total deflection under the dead load, and Δ is the total deflection under the dead load plus live load. The difference between Δ_D and Δ represents the effect which the live load has on the deflection.

Determination of δ_w.—The necessary computations are recorded in Table 3, and no further explanations are required. Of course, it would not have been necessary to compute the equivalent concentrated angle changes, since there were no discontinuities in the $\frac{M}{E\,I}$-diagram, and the simplified method mentioned in Part II of the paper could have been used.

Determination of δ_a.—This component of the deflection is the upward movement caused by the moment that the axial dead load has about the neutral

TABLE 3.—Determination of Downward Deflection Caused by Bending Moment Due to Weight of Boom (δ_w)

Line	Description	Tenth Points: 0	1	2	3	4	5	6	7	8	9	10	Common factor
	Moment of inertia I (in.⁴)	42,057	5,363	7,098	11,834	11,834	11,834	11,834	11,834	7,098	3,387	5,071	....
							10 λ = 125 Ft						....
1	Moment due to weight of boom	0	1.888	3.313	4.313	4.777	4.925	4.741	4.237	3.188	1.760	0	10^6
2	Distributed angle change	0	351	469	365	404	416	400	358	449	520	0	$1/E$
3	Concentrated angle change		312.2	432.0	388.8	399.5	411.3	395.7	380.2	445.7	421.5		λ/E
4	Assumed average slope		1,600	1,287.8	855.8	467.0	67.5	−343.8	−739.5	−1,119.7	−1,565.4	−1,986.9	λ/E
5	Trial deflection		1,600	2,887.8	3,743.6	4,210.6	4,278.1	3,934.3	3,194.8	2,075.1	509.7	−1,477.2	λ^2/E
6	Linear correction to deflection		+147.7	295.4	443.1	590.8	738.5	886.2	1,033.9	1,181.6	1,329.3	+1,477.2	λ^2/E
7	Resultant deflection	0	1,747.7	3,183.2	4,186.7	4,801.4	5,016.6	4,820.5	4,228.7	3,256.7	1,839.0	0	λ^2/E
8	Resultant deflection in inches = δ_w	0	1.31	2.39	3.14	3.60	3.77	3.62	3.17	2.44	1.38	0	

TABLE 4.—Determination of δ_a and Δ_D

Line	Description	Tenth Points: 0	1	2	3	4	5	6	7	8	9	10	Common factor
							10 λ = 125 Ft						
	Moment of inertia I (in.⁴)	42,057	5,363	7,098	11,834	11,834	11,834	11,834	11,834	7,098	3,387	5,071	
1	Assumed deflection due to axial dead load, δ_a	0	−0.02	−0.03	−0.04	−0.04	−0.03	−0.03	−0.03	−0.03	−0.02	0	
2	Moment arm $[\delta_w - (x + y) - \delta_a]$	+6.33	−1.70	−1.56	−1.27	−0.63	−0.24	−0.19	−0.43	−0.26	−0.60	0	
3	Moment (51,250 times moment arm)	+325	−87.2	−80.0	−65.2	−32.3	−12.3	−9.75	−22.1	−13.3	−30.7	0	10^3
4	Distributed angle change	−7.71	+16.25	+11.26	+5.50	+2.72	+1.04	+0.83	+1.87	+1.87	+9.08	0	$1/E$
5	Assumed average slope		−40.00	−23.75	−12.49	−6.99	−4.27	−3.23	−2.40	−0.53	+1.34	+10.42	λ/E
6	Trial deflection (1)	+82.90	+42.90	+19.15	+6.66	−0.33	−4.60	−7.83	−10.23	−11.76	−10.42	0	λ^2/E
7	Trial deflection (2)	−0.64	+1.35	+0.94	+0.46	+0.22	+0.09	+0.07	+1.56	+1.56	+0.76	0	λ^2/E
8	Linear correction to deflection	−82.26	−74.03	−65.80	−57.58	−49.35	−41.13	−32.90	−24.68	−16.45	−8.22	0	λ^2/E
9	Resultant deflection, δ_a	0	−29.78	−45.71	−50.46	−49.46	−45.64	−40.66	−33.35	−26.65	−17.90	0	λ^2/E
10	Resultant deflection, δ_a	0	−0.0223	−0.0342	−0.0378	−0.0371	−0.0342	−0.0305	−0.0250	−0.0200	−0.0134	0	
	Total Dead Load Deflection $\Delta_D = \delta_w + \delta_a$ (Line 8, Table 3 + Line 10, Table 4)												
11	....	0	1.29	2.36	3.10	3.56	3.74	3.59	3.14	2.44	1.37	0	

TABLE 5.—Determination of δ and Δ.

Line	Description	Tenth Points: 0	1	2	3	4	5	6	7	8	9	10	Common factor
	Moment of inertia I (in.⁴)	42,057	5,363	7,098	11,834	11,834	11,834	11,834	11,834	7,098	3,387	5,071	
							10 λ = 125 Ft						
1	Assumed deflection due to axial D.L. & L.L., δ	0	−0.05	−0.08	−0.09	−0.09	−0.08	−0.08	−0.06	−0.05	−0.03	0	
2	Moment arm $[\delta_w - (x + y) - \delta]$	+6.33	−1.73	−1.61	−1.32	−0.68	−0.29	−0.24	−0.46	−0.29	−0.62	0	
3	Moment [(51,250 + 64,950) × moment arm]	+735	−201	−187	−154	−79.0	−33.5	−28.0	−53.5	−34.0	−72.1	0	10^3
4	Distributed angle change	−17.5	+37.5	+26.4	+13.0	+6.68	+2.83	+2.37	+4.52	+4.79	+21.3	0	$1/E$
5	Assumed average slope		−50.00	−12.5	+13.9	+26.9	+33.58	+36.41	+38.78	+43.30	+48.09	+69.39	λ/E
6	Trial deflection (1)	−247.85	−297.85	−310.35	−296.45	−269.55	−235.97	−199.56	−160.78	−117.48	−69.39	0	λ^2/E
7	Trial deflection (2)	−1.46	+3.13	+2.20	+1.08	+0.56	+0.24	+0.20	+0.38	+0.40	+1.77	0	λ^2/E
8	Linear correction to deflection	+249.31	+224.37	+199.44	+174.51	+149.58	+124.65	+99.72	+74.79	+49.86	+24.93	0	λ^2/E
9	Resultant deflection, δ	0	−70.35	−108.71	−120.86	−119.41	−111.08	−99.64	−85.61	−67.22	−42.69	0	λ^2/E
10	Resultant deflection, δ	0	−0.0528	−0.0814	−0.0906	−0.0895	−0.0834	−0.0746	−0.0642	−0.0505	−0.0321	0	
	Total Dead and Live Load Deflection $\Delta = \delta_w + \delta$ (Line 8, Table 3 + Line 10, Table 5)												
11	....	0	1.26	2.31	3.05	3.51	3.69	3.54	3.11	2.39	1.35	0	

axis. That this movement is upward can be ascertained from the fact that, except at the left end, the line of action of the axial load is below the neutral axis. The moment arm is $\delta_w - (x + y)$ prior to the upward deflection δ_a, for which assumed values are given in line 1 of Table 4. The final moment arm is then $\delta_w - (x + y) - \delta_a$. In the case of point 5λ, for example, this is $3.77 - (1.03 + 2.95) - 0.03 = -0.24$. If the results do not coincide with the assumed values of δ_a, the procedure is repeated with corrected values. At the bottom of Table 4, in line 11, the deflection $\Delta_D = \delta_w + \delta_a$ of the boom under dead load is given.

Determination of δ.—This is the deflection due to both dead and live axial loads. Its computation follows the same pattern as that of δ_a. The force acting is 51,250 + 64,950 = 116,200 lb, and a set of values of δ is assumed and given in line 1 of Table 5. Again, if the results do not coincide with the assumed values of δ, corrected values are substituted. The values of Δ (which is the deflection of the boom due to live and dead load) are given in line 11, Table 5. Comparison with corresponding values of Δ_D shows that the live load reduces the deflection by a small amount, namely $\Delta_D - \Delta$.

ALFRED S. NILES,[25] ASSOC. M. AM. SOC. C. E.—The method of computation described in this paper is very ingenious, and should prove to be a great timesaver in the solution of many types of problems. Although the author has shown applications to both beams and columns of single span, he has failed to warn the reader that his method is not directly applicable to continuous members, for which the bending moments over the supports must be obtained by the use of the three-moment equation, the method of moment distribution, or an equivalent method. Perhaps the most awkward member of this type, from the point of view of the stress analyst, is a continuous beam of nonuniform section that is subjected to combined bending and compression. This is handled most readily by the method of moment distribution, proper allowance being made for the effect of the axial load when computing fixed-end moments and the carry-over and stiffness factors. The computation of these quantities for members of nonuniform section by previously published methods is a slow and tedious procedure. The methods proposed by the author appear more convenient for this work than any other that has yet been suggested, including that of the writer and J. S. Newell.[7]

In his numerical examples the author divides the beam into segments of equal length. This greatly simplifies the work, and is nearly always allowable when the transverse load (or "angle change") can be completely represented by a smooth curve. If this is not allowable, however, as when unequally spaced concentrated loads are present, much of the advantage is lost. In fact, the author's method becomes practically the same as that described,[26] the only differences being in the method of recording the computations, and in the more accurate, although more time-consuming, method of allowing for the curvature of the loading (or "angle change") diagram, which the writer treats as composed of straight segments. In most practical problems, however, the value of the greater accuracy of the author's method of allowing for this factor is questionable.

In computing "equivalent concentrated angle changes," the author uses formulas from Figs. 3 and 5 that require division by 6, 12, or 24, depending on the shape of the angle-change curve. It would seem simpler to include the factor 6, 12, or 24 in the "common factor" by which the moments or deflections are to be multiplied to get the final results. This would result in most of the values in the tabulated computations being 6, 12, or 24 times as large as those obtained in the method as described, but would not affect the final results. Care would have to be taken, however, to use the same factor throughout the span when part of the loading curve was straight and part curved, but in any given problem it would be easy to multiply the formulas of Figs. 3 and 5 by $\frac{2}{2}$ or $\frac{4}{4}$ in order to obtain a common denominator for all the formulas used.

In some of his examples (as in Fig. 1(*d*)) the author computes the actual value of the shear at the left end of the beam before computing the shears at other points, whereas in others (as in Fig. 1(*e*)) he starts the shear computations from an arbitrarily assumed figure and makes a final correction to the bending moments, if necessary. The writer has two objections to the latter practice, although it seems to be preferred by the author. The first is that it is often necessary to know the shears at various points along the span, and, in the latter practice, it would be necessary to remember to correct the values originally found to obtain the true ones. This could be done easily, and the objection would be unimportant if it stood alone. The more serious objection is that the practice eliminates a valuable internal check on the computations. If the actual shear at the left end is first computed, then the moment at the right end, computed by summation of the shears along the span, should be the same as that stated in the formulation of the problem. If the two values are not in substantial agreement, an error has been made. In the method of Fig. 1(*e*), one does not know whether the necessary moment correction is due solely to the difference between the assumed and actual shears at the left end, or whether it is partly due to a numerical error of computation. Since the actual shear at the left end can be computed quite easily, the check obtained justifies the little additional work involved in using it.

The writer notes that the author has reversed the usual convention and considers that loads are positive when they act upward. He heartily indorses this practice. He wonders, however, why the same change was not made in the conventions for slope and deflection. That would have involved the elimination of the minus sign from the definition of "angle change" as $-\frac{M}{E\,I}$; but that sign is not essential. It is there only to reconcile some independently assumed conventions which proved to be lacking in logical consistency. It is really much simpler to assume upward loads and deflections as positive. Then one can differentiate the equation of the elastic curve four times, obtaining successively the slope, bending moment, shear, and loading, without having to remember to reverse signs arbitrarily at various steps.

[25] Prof., Aeronautic Eng., Leland Stanford Junior Univ., Aero. Laboratory, Stanford Univ., Stanford University, Calif.

[26] "Airplane Structures," by A. S. Niles and J. S. Newell, 2d Ed., New York, N. Y., 1938, Vol. I, pp. 58–60.

A related point is in connection with nomenclature. In his paper the author terms the quantity, $-\frac{M}{E\,I}$, the "angle change." Actually, as he states, he is using that expression as an allowable approximation for what mathematicians call the "curvature," $\frac{1}{r}$, in which r is the radius of curvature. Since the mathematicians already have given the quantity a name, why was it necessary to rechristen it? It might be considered awkward to speak of a "curvature curve," although the expression should be quite as clear as "angle-change curve"; and "concentrated curvature" should be as clear as "concentrated angle change." What the author has termed "angle change" is really "rate of slope change," and the latter term would really be preferable to the former, if "curvature" is to be replaced by something else.

In studying the numerical examples, the writer was unable to verify one of the author's figures. In Fig. 15(*a*) the equivalent concentrated angle change at the section of change in moment of inertia is shown as −404.90. This appears to be a quantity to be obtained by use of the formulas of Fig. 5(*a*), assuming the distributed angle-change curves produced to have ordinates either one tenth of, or ten times, those of the actual curve, in the adjacent segments of the beam. On this basis, the concentrated angle change in question would appear to be $-(3 \times 513.5 + 10 \times 803.6 - 1 \times 911.0 + 3 \times 91.10 + 10 \times 80.36 - 1 \times 51.35) \times \frac{1}{24} = -403.79$. The actual difference between this value and the author's is of no practical consequence, but it would be interesting to learn whether the figure in the text was computed by some other method.

Although the author's paper is subject to the foregoing minor criticisms, he deserves much credit for developing a valuable new tool for the use of the structural engineer.

Camillo Weiss,[27] M. Am. Soc. C. E.—The method outlined will undoubtedly be found useful in many types of problems other than those discussed by Professor Newmark, and the determination of ordinates to influence lines is one of these. It is readily applicable because influence lines can be considered as ratios between corresponding deformations. Furthermore, because only ratios are required, the various "common factors" may be disregarded, and scales may be adopted and changed to suit convenience at any step in the consecutive computations, provided relative scales remain the same. The moment diagrams are bounded by straight lines; therefore the results are accurate for straight-line or parabolic variations of moments of inertia. For other variations satisfactory approximations may be obtained.

The writer has computed influence lines for three typical cases, shown in Figs. 33, 34, and 35, and a study of these calculations will show readily the relative ease of the work required. The conventional calculation methods involve the same steps, but by applying the author's method the amount of laborious arithmetical work is greatly reduced.

[27] Designer, Bethlehem Steel Co., Fabricated Steel Constr., Eng. Dept., Bethlehem, Pa.

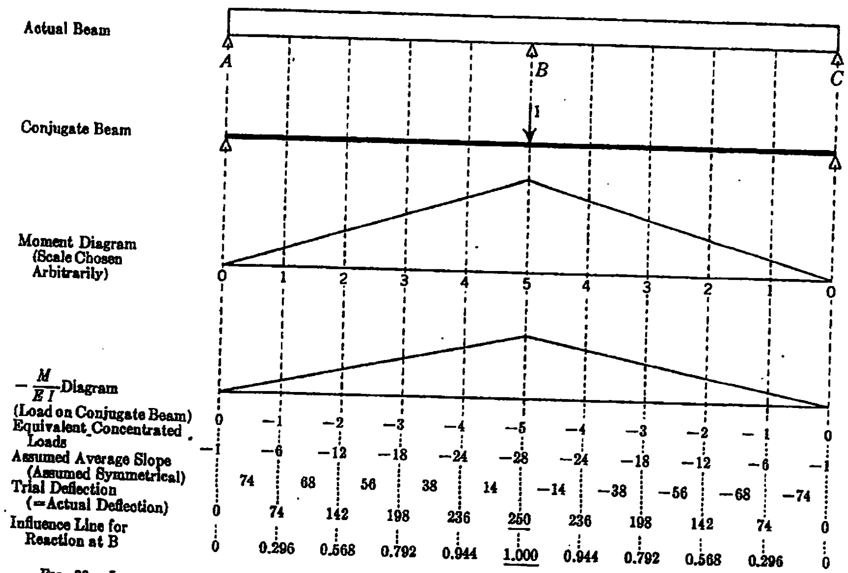


Fig. 33.—Influence Line for Reaction at Center Support of Beam Continuous Over Three Supports (Spans Equal; Moment of Inertia Constant)

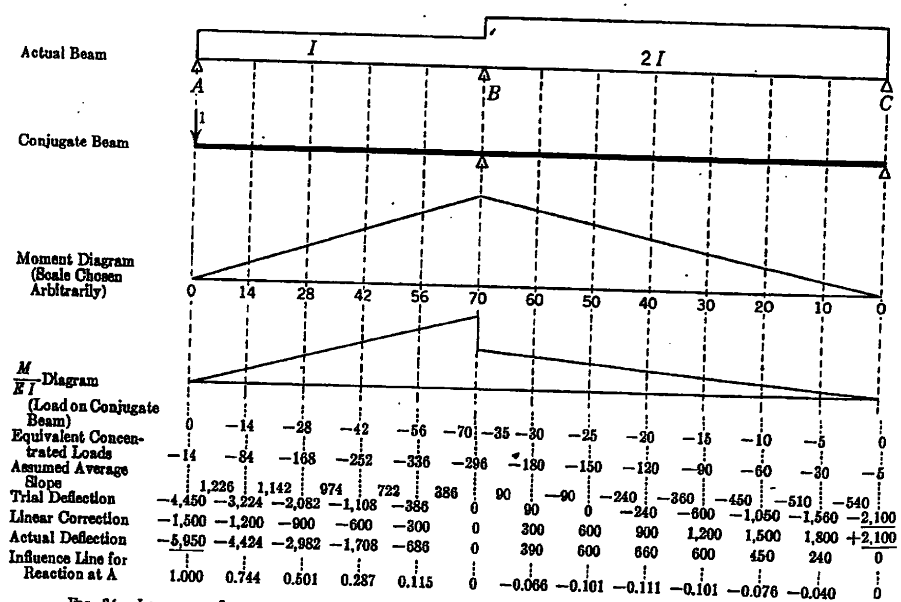


Fig. 34.—Influence Line for Reaction at End Support A of Beam Continuous Over Three Supports (Spans Unequal; Moments of Inertia Differ for Spans But Are Constant Within Span Length)

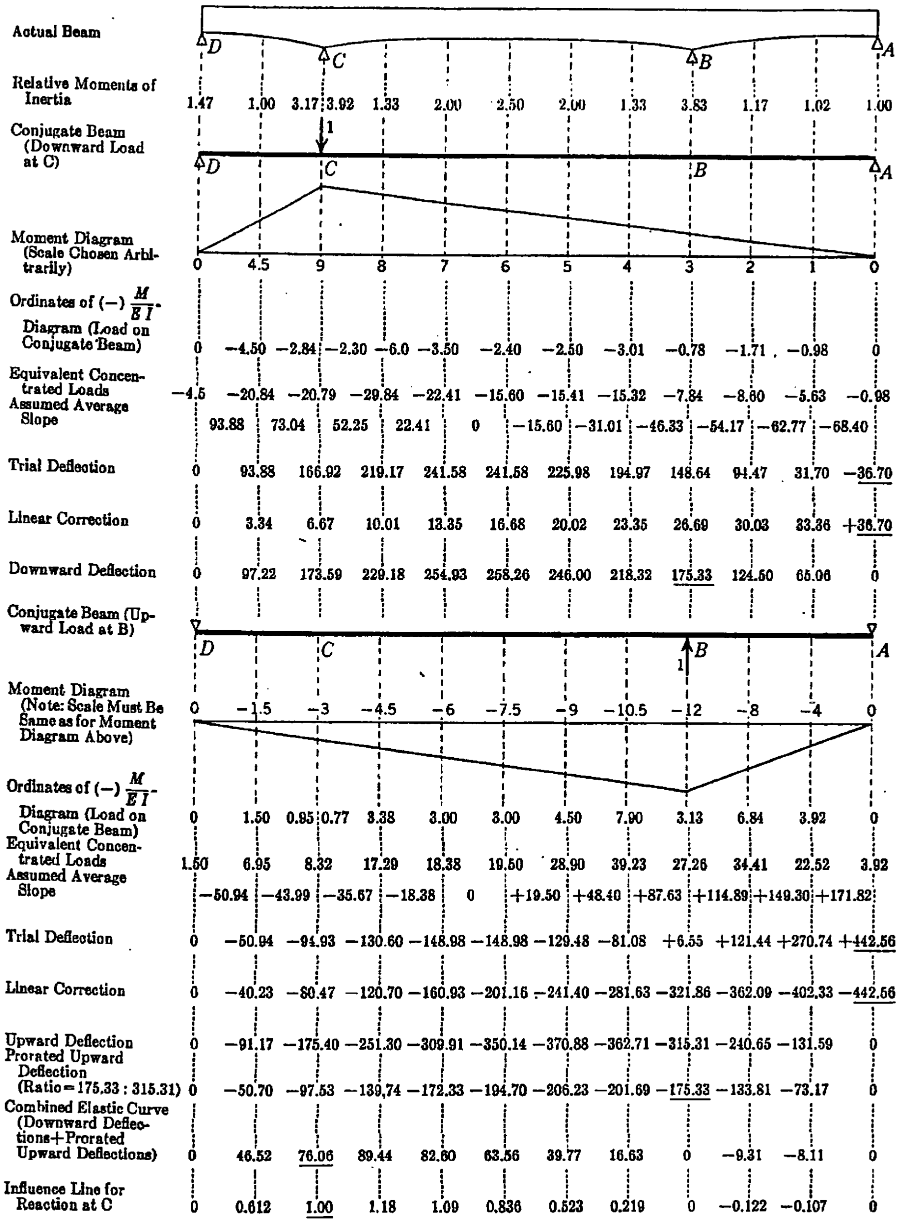


FIG. 35.—INFLUENCE LINE FOR REACTION AT SUPPORT C OF A BEAM CONTINUOUS OVER FOUR SUPPORTS (SPANS UNEQUAL; MOMENTS OF INERTIA VARIABLE; THE VARIATION ASSUMED TO BE OF LINEAR DIMENSION)

The influence diagram in Fig. 35 is obtained as the algebraic sum of two elastic curves of the simple beam AD. The first elastic curve is for a unit load at C acting downward. The second elastic curve is for a concentrated load at B chosen to nullify the deflection at B. This is accomplished by prorating the deflections due to a unit load at B acting upward. The procedure may be extended to any number of supports. It may be of interest to note that for a structure comparable to that of Fig. 35, but with vertical columns continuous at B and C and hinged at their bases, conventional methods furnished the following influence ordinates: 0.000, + 0.617, + 1.00, + 1.16, + 1.04, + 0.775, + 0.448, + 0.161, 0.000, − 0.090, − 0.077, 0.000.

The constantly increasing number of indeterminate structures which are being built has made it essential for the designer to familiarize himself with the method of moment distribution devised by Hardy Cross,[28] M. Am. Soc. C. E. The author's method will provide great assistance in the determination of stiffness and carry-over factors, and in other less obvious ways.

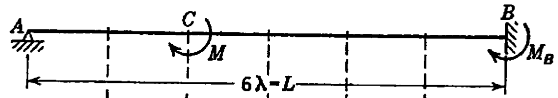
Professor Newmark is to be congratulated for having produced a useful and well-presented paper, which is a definite contribution to engineering design methods.

A. A. EREMIN,[29] ASSOC. M. AM. SOC. C. E.—An interesting method of making successive approximations for the computation of stresses and deformations is described in this paper. The method of successive approximations is exceedingly useful when sections of members carrying loads vary along the span length and when the sectional variation is difficult to express by a simple mathematical formula. The problems solved by the author clarify the practical value of the method.

A useful addition to the cases considered by Professor Newmark might be the case of a member resisting a bending moment applied at an intermediate section between the supported ends. This case may occur in the column that receives load applied through a bracket. Applying the method of successive approximations, the stresses and deformations in member AB, loaded with a bending moment M at section C, were computed as shown in Fig. 36. The member was divided into six sections. (Figure 36 follows on page 1212)

[28] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

[29] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.



(a) Deflection Due to Moment M_B = 100 at Right End

Moments	0		17		33		50		67		83		100
Distributed Angle Changes			−17		−33		−50		−67		−83		−47
Assumed Average Slope, Part (1)		−17		0		33		83		150		233	
Trial Deflection, Parts (1) and (2)	482		499		499		466		383		233		
Linear Correction to Deflection	482		402		320		240		160		80		0
Resultant Deflection	0		97		179		226		223		153		0

$$\text{Slope at right end} = (153 + 47)\frac{\lambda}{EI} = 200\frac{\lambda}{EI}$$

(b) Deflection Due to Moment M = 100

Moments	0		−17		−33 ¦ +67		+50		+33		+17		0
Distributed Angle Changes			+17		−17		−50		−33		−17		−3
Assumed Average Slope, Part (1)		−37		−54		−37		+13		+46		+63	
Trial Deflection, Parts (1) and (2)	−6		31		85		122		109		63		
Linear Correction to Deflection	6		5		4		3		2		1		
Resultant Deflection	0		36		89		125		111		64		

$$\text{Slope at right end} = (64 + 3)\frac{\lambda}{EI} = 67\frac{\lambda}{EI}$$

$$\text{Moment at left end} = M_B = \frac{67 \times 100}{200} = 33$$

Fig. 36.—Restraining Moment Produced by Moment Applied at a Section Between Beam Supports

Myron L. Gossard,[30] Jun. Am. Soc. C. E.—The numerical procedure for finding deflections and slopes of beams combines fundamental concepts of beam elasticity and geometry, simplicity and accuracy of method, and clearness as to beam action under load. These qualities, plus the fact that the procedure follows closely that of computing shear and moment diagrams, certainly will make it valuable to both structural engineers and engineering students. The application of the method is particularly effective in the analysis of continuous frameworks where it is necessary to determine certain beam constants and load constants. The author's procedure seems to possess all the advantages of the method of the column analogy when applied to beams, and gives a clearer picture of the beam action.[31] Also, the column analogy does not give the deflections directly as does the numerical procedure.

The writer has used a special application of the fundamental Newmark method to arrive at a procedure which, it is believed, is somewhat more convenient for finding stiffnesses, carry-over factors, fixed-end moments, and deflections of members of continuous frames—especially unsymmetrical members

[30] Stress Analyst, Airplane Div., Curtiss-Wright Corp., Louisville, Ky.

[31] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y., 1932, pp. 46–47.

that may be subdivided into segments that can be approximated by "standard" beam forms. The "standard" beam forms are those shown in Fig. 37, where the haunch curves may be either straight lines, half quadratic parabolas, or

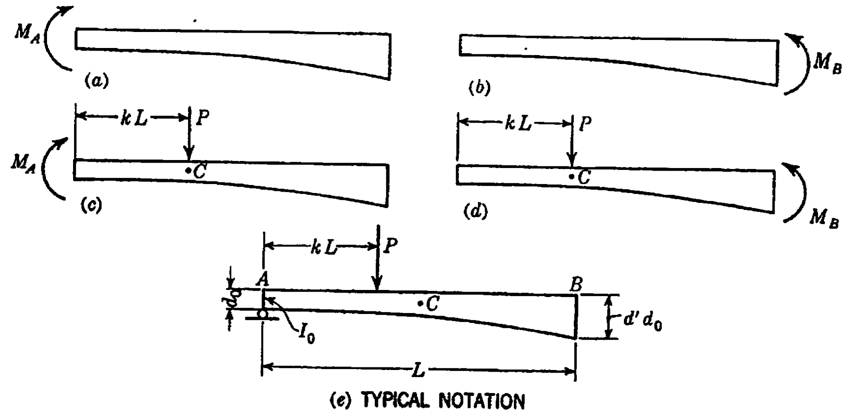


Fig. 37.—Standard Beam Forms (Beam Outlines Only), with Load Types

half cubic parabolas, with vertexes at the shallow end of the beam. Following Professor Newmark's numerical procedure, sets of curves similar to Figs. 38

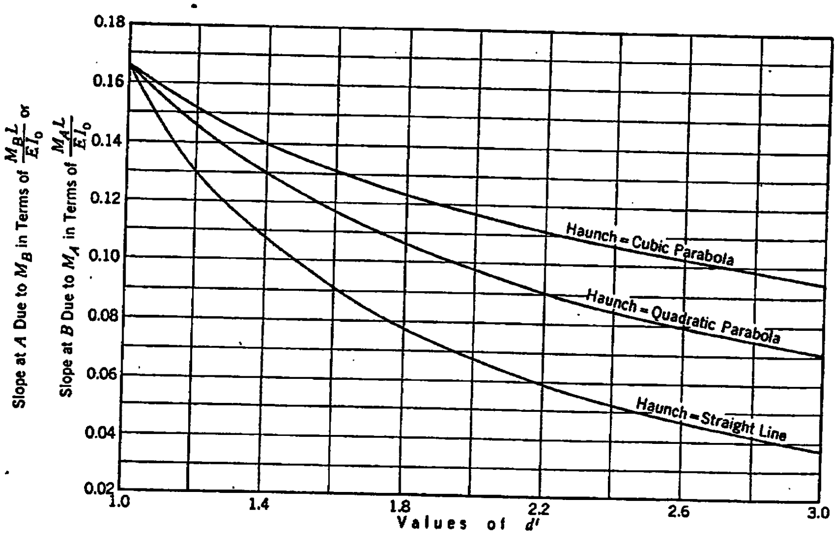


Fig. 38.—Slope at One End of a Beam Due to a Moment Applied at the Other End (See Figs. 37(a) and 37(b))

and 39 were drawn for each case, using twelve equal divisions of beam length, and based on the assumption that, at any section, the moment of inertia varies as the $\frac{5}{2}$ power of the beam depth. This assumption seems to the writer to be valid and sufficiently accurate for both structural steel (plate girder or I-beam) and reinforced concrete (rectangular or T-beam) construction, inasmuch as the exponent must lie between 2 and 3 for these types and some error in this respect does not appreciably affect the results of analyses.[32] Fig. 38 is for the case of a moment applied at one end or the other end of a beam (Figs. 37(*a*) and 37(*b*)),

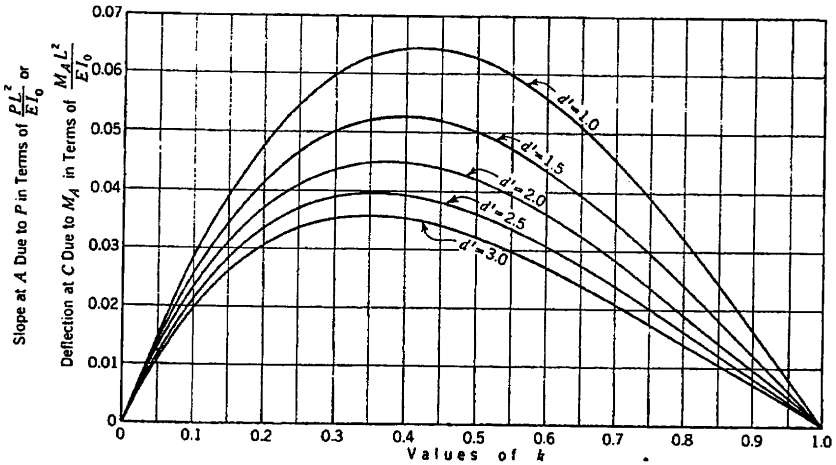


FIG. 39.—SLOPE AT END A DUE TO A CONCENTRATED LOAD (See Fig. 37(*c*)); OR DEFLECTIONS ALONG THE BEAM DUE TO A MOMENT APPLIED AT END A (See Fig. 37(*a*)) (Haunch = Quadratic Parabola)

giving the slope at A due to M_B, in terms of $\frac{M_B L}{E I_o}$ (or the slope at B due to M_A, in terms of $\frac{M_A L}{E I_o}$). For the purposes of a complete solution, similar curves (not published) were constructed for the slope at A due to M_A, and for the slope at B due to M_B. Fig. 39 is typical of curves giving the slope at A due to a concentrated load P (Fig. 37(*c*)), in terms of $\frac{P L^2}{E I_o}$. The curves of Fig. 39 also give the deflections along the beam due to M_A (Fig. 37(*a*)), in terms of $\frac{M_A L^2}{E I_o}$, by Maxwell's law of reciprocal displacements. Curves similar to those of Fig. 39 (not published) were prepared for straight-line and third-degree parabolic haunches; and for the slope at B due to a concentrated load P (Fig. 37(*c*)), the latter set of curves also giving the deflections along the beam due to M_B (Fig. 37(*b*)). Briefly, six sets of curves are needed for a complete solution of a case of the type in Fig. 40, representing:

[32] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y., 1932, pp. 3-5 and 169-171.

Case	Description
(*a*)	The slope at A due to M_A
(*b*)	The slope at B due to M_A, which is equal to the slope at A due to M_B
(*c*)	The slope at B due to M_B
(*d*)	The deflection at C due to M_A, which is equal to the slope at A due to a concentrated load P at C
(*e*)	The deflection at C due to M_B, which is equal to the slope at B due to a concentrated load P at C
(*f*)	The deflection at beam-center due to a concentrated load P

An example of what may be encountered and what may be done with the aid of such data is illustrated in Fig. 40, from which all the information necessary for a complete analysis of the beam as a part of a continuous frame is derived. However, if deflection is not important or if "highly accurate" deflection curves and influence lines are not required, a considerable amount of the work can be eliminated. Fig. 40(*a*) shows the beam to be made up of three elastic segments, L_1, L_2, and L_3, with an assumed inelastic segment at each end. The trapezoidal moment diagram for each elastic segment as a part of the whole is divided into two triangular diagrams; then end slopes and deflections for each segment due to each of its triangular moment diagrams are taken from the appropriate curves of the types of Figs. 38 and 39. The end slopes are added at each segmental junction or "joint" to become the "equivalent concentrated angle changes" from which the average slopes and string deflections of the joints are obtained as in the fundamental procedure. To the string deflections are added the segmental deflections at selected points between the joints (at center points in the example) to obtain the deflection curve. Figs. 40(*b*) and 40(*c*) show the calculations and deflection curves for $M_A = 100$ and $M_B = 100$, respectively. In Fig. 40(*d*) the simple beam deflection curves are combined to give curves of deflection for end moments of 100 with the far ends fixed These may be used as influence lines for fixed-end moments. Fig. 40(*e*) is included to illustrate the procedure for finding the influence line for beam-center deflection which is always close to the maximum deflection in both simple or continuous beams. By proper combinations of Figs. 40(*b*), 40(*c*), and 40(*e*) the center deflection for any load condition on the continuous beam may be found.

To complete the discussion there follow the calculations for moment-distribution constants, fixed-end moments, and simple beam center deflection for dead and live loads shown in Fig. 40(*f*). In the calculations involving influence lines and distributed loads, the area under a curve is approximated by finding the area under a parabola passing through three points, which area (A) is given by

$$A = \frac{\lambda}{6}(a + 4b + c) \qquad (33)$$

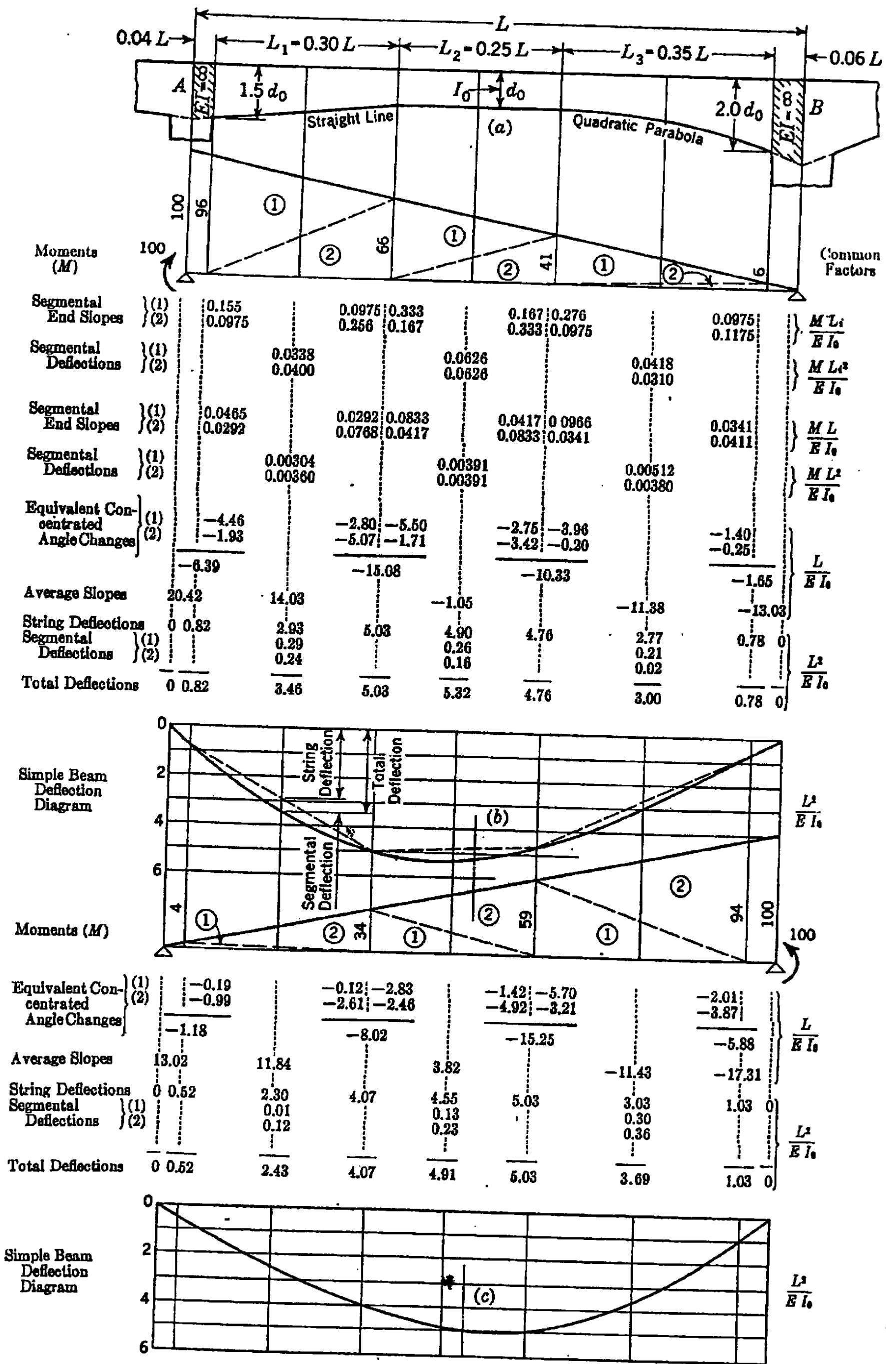


FIG. 40.—COMPUTATIONS FOR AN UNSYMMETRICAL BEAM

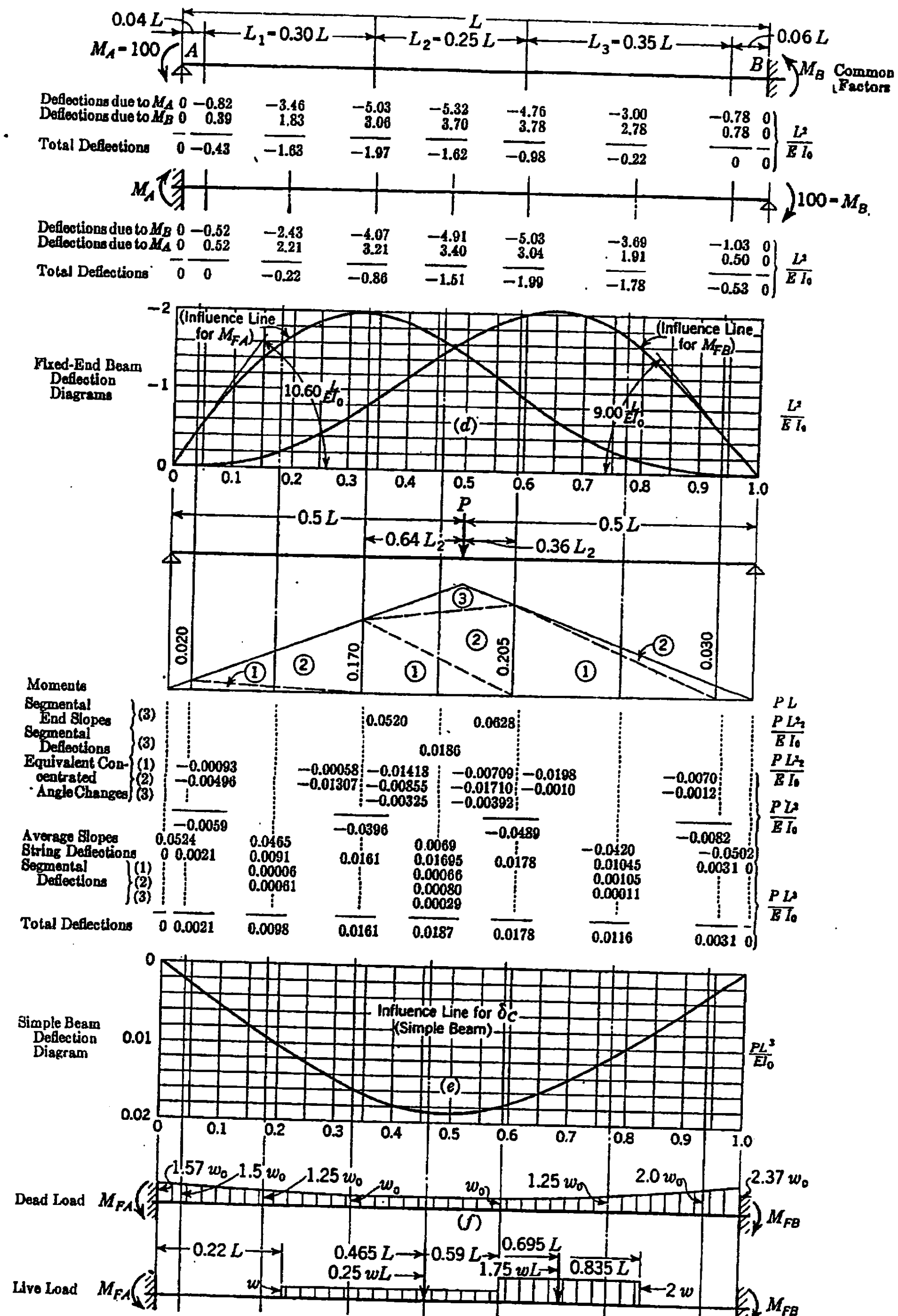


FIG. 40.—COMPUTATIONS FOR AN UNSYMMETRICAL BEAM (Continued)

In Eq. 33, a and c are bounding ordinates of the horizontal length λ, and b is the center ordinate.

The beam constants are as follows:

Assuming positive rotations as clockwise, the rotation at one end of a simple beam, due to unit clockwise moment applied at that end or the opposite end, is—

$$\phi_{AA} = +\,0.2042\,\frac{L}{E\,I_o} \quad \text{(34a)}$$

$$\phi_{BA} = -\,0.1303\,\frac{L}{E\,I_o} \quad \text{(34b)}$$

$$\phi_{BB} = +\,0.1731\,\frac{L}{E\,I_o} \quad \text{(34c)}$$

and

$$\phi_{AB} = -\,0.1302\,\frac{L}{E\,I_o} \quad \text{(34d)}$$

In Eqs. 34, the second letter of the subscript denotes the end of the beam where the moment is applied, Eqs. 34*a* and 34*b* referring to Fig. 40(*b*) and Eqs. 34*c* and 34*d* to Fig. 40(*c*).

Assuming that positive moments produce tension in the bottom fibers of a beam, the carry-over factor is the ratio of the moment at one end to the moment applied at the other end, when one end or the other is fixed:

$$r_{AB} = \frac{\phi_{AB}}{\phi_{BB}} = \frac{-0.1302}{0.1731} = -\,0.752 \quad \text{(35a)}$$

and

$$r_{BA} = \frac{\phi_{AB}}{\phi_{AA}} = \frac{-0.1302}{0.2042} = -\,0.637 \quad \text{(35b)}$$

For example, Eq. 35*a* yields the carry-over factor at end A when end B is fixed, and Eq. 35*b*, the carry-over factor at end B when end A is fixed. For the moment stiffness S_M, the moment at end A required to produce unit rotation at end A when end B is fixed, is

$$(S_M)_A = \frac{1}{\phi_{AA} - r_{AB}\,\phi_{AB}} = 9.40\,\frac{E\,I_o}{L} \quad \text{(36a)}$$

Similarly for the moment at end B required to produce unit rotation at end B when end A is fixed—

$$(S_M)_B = \frac{1}{\phi_{BB} - r_{BA}\,\phi_{AB}} = 11.1\,\frac{E\,I_o}{L} \quad \text{(36b)}$$

From Figs. 40(*d*) and 40(*f*), the fixed-end moments (M_F) are computed as follows.

Dead Load.—

$$\frac{0.30\,L}{6}\,[1.5\,(0.43) + 4\,(1.25)\,(1.63) + 1.97]\,w_o = 0.538\,w_o\,L$$

$$\frac{0.25\,L}{6}\,[1.97 + 4\,(1.62) + 0.98]\,w_o = 0.393\,w_o\,L$$

$$\frac{0.35\,L}{6}\,[0.98 + 4\,(1.25)\,(0.22) + 0]\,w_o = 0.121\,w_o\,L$$

$$0.04\,L\,(1.53)\,(0.22)\,w_o = 0.013\,w_o\,L$$

$$\text{Total} = 1.065\,w_o\,L$$

$$(M_F)_A = 1.065\,w_o\,L \times \frac{L}{10.60} = 0.100\,w_o\,L^2\,(-) \quad \text{(37a)}$$

and

$$\frac{0.30\,L}{6}\,[0 + 4\,(1.25)\,(0.22) + 0.86]\,w_o = 0.098\,w_o\,L$$

$$\frac{0.25\,L}{6}\,[0.86 + 4\,(1.51) + 1.99]\,w_o = 0.371\,w_o\,L$$

$$\frac{0.35\,L}{6}\,[1.99 + 4\,(1.25)\,(1.78) + 2\,(0.53)]\,w_o = 0.697\,w_o\,L$$

$$0.06\,L\,(2.18)\,(0.27)\,w_o = 0.035\,w_o\,L$$

$$\text{Total} = 1.201\,w_o\,L$$

$$(M_F)_B = 1.201\,w_o\,L \times \frac{L}{9.00} = 0.134\,w_o\,L^2\,(-) \quad \text{(37b)}$$

Live Load.—

$$\frac{0.37\,L}{6}\,[1.75 + 4\,(1.83) + 0.98]\,w = 0.620\,w\,L$$

$$\frac{0.245\,L}{6}\,[0.98 + 4\,(0.40) + 0.05]\,2\,w = 0.215\,w\,L$$

$$0.25\,w\,L\,(1.62) = 0.405\,w\,L$$

$$1.75\,w\,L\,(0.46) = 0.805\,w\,L$$

$$\text{Total} = 2.045\,w\,L$$

$$(M_F)_A = 2.045\,w\,L \times \frac{L}{10.60} = 0.193\,w\,L^2\,(-) \quad \text{(38a)}$$

and

$$\frac{0.37\,L}{6}\,[0.31 + 4\,(1.18) + 1.99]\,w = 0.433\,w\,L$$

$$\frac{0.245\,L}{6}\,[1.99 + 4\,(1.96) + 1.34]\,2\,w = 0.913\,w\,L$$

$$0.25\,w\,L\,(1.51) = 0.378\,w\,L$$

$$1.75\,w\,L\,(2.0) = 3.50\,w\,L$$

$$\text{Total} = 5.224\,w\,L$$

$$(M_F)_B = 5.224\,w\,L \times \frac{L}{9.00} = 0.580\,w\,L^2\,(-) \quad \text{(38b)}$$

From Figs. 40(*e*) and 40(*f*) the simple beam center deflections are computed as follows.

Dead Load.—

$$\frac{0.30\ L}{6}\ [1.5\ (0.0021) + 4\ (1.25)\ (0.0098) + 0.0161]\ w_o = 0.00341\ w_o\ L$$

$$\frac{0.25\ L}{6}\ [0.0161 + 4\ (0.0187) + 0.0178]\ w_o = 0.00453\ w_o\ L$$

$$\frac{0.35\ L}{6}\ [0.0178 + 4\ (1.25)\ (0.0116) + 2\ (0.0031)]\ w_o = 0.00478\ w_o\ L$$

$$0.04\ L\ (1.53)\ (0.0011)\ w_o = 0.00007\ w_o\ L$$

$$0.06\ L\ (2.18)\ (0.0016)\ w_o = 0.00021\ w_o\ L$$

$$\text{Total} = 0.01300\ w_o\ L$$

$$\delta_c = 0.01300\ w_o\ L \times \frac{L^3}{E\ I_o} = 0.0130\ \frac{w_o\ L^4}{E\ I_o}\ (+) \qquad (39)$$

Live Load.—

$$\frac{0.37\ L}{6}\ [0.0112 + 4\ (0.0179) + 0.0178]\ w = 0.00620\ w\ L$$

$$\frac{0.245\ L}{6}\ [0.0178 + 4\ (0.0138) + 0.008]\ 2\ w = 0.00662\ w\ L$$

$$0.25\ w\ L\ (0.0187) = 0.00467\ w\ L$$

$$1.75\ w\ L\ (0.0145) = 0.0254\ w\ L$$

$$\text{Total} = 0.0429\ w\ L$$

$$\delta_c = 0.0429\ w\ L \times \frac{L^3}{E\ I_o} = 0.0429\ \frac{w\ L^4}{E\ I_o}\ (+) \qquad (40)$$

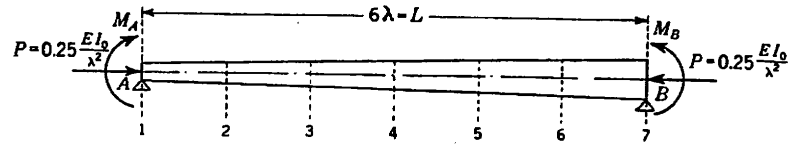
Lack of space prevents the inclusion of the various curves prepared for use in the discussion of Fig. 40. In other respects, however, the writer has entered into more detail than is necessary in most applications of the numerical procedure because the emphasis of the Newmark paper is on a technique. Technique may rightfully assume prominence in problems such as this.

ROBERT A. WILLIAMSON,[33] JUN. AM. SOC. C. E.—In members subjected to combined axial and bending loads (commonly called beam columns) the effect of secondary moments caused by end thrusts cannot be ignored safely when the compressive load is any appreciable percentage of the critical buckling load. This is plainly demonstrated by the results of Fig. 17.

When the beam column is one of a series of members comprising a continuous structure, the values of the elastic constants and fixed-end moments required for the usual moment-distribution analysis depend, in part, on the magnitude and sign of the axial load.

For the case of variable section, evaluation of these quantities is greatly facilitated by the use of Professor Newmark's procedure, details of which are shown in Figs. 41 to 44, inclusive, using as an example the beam of Figs. 10 and 11 subjected to an axial compressive load.

[33] Stress Analyst, Vega Aircraft Corp., Burbank, Calif.



I/I_o

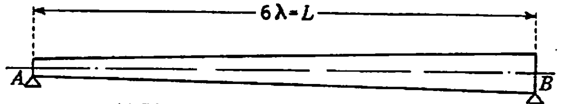
(*a*) Deflections and End Slopes for $M_A = 180$, $P = 0.25\ \frac{E\ I_o}{\lambda^2}$

	1		2		3		4		5		6		7	Common Factors
w_i for $M_A = 180$ From Fig. 10(*c*)	0		110.6		141.4		130.8		97.1		51.1		0	$\lambda^2/(E\ I_o)$
Assumed w_a	0		34.0		47.0		44.0		36.0		19.0		0	$\lambda^2/(E\ I_o)$
$w_o = w_i + w_a$	0		144.6		188.4		174.8		133.1		70.1		0	$\lambda^2/(E\ I_o)$
Moments due to P	0		36.2		47.2		43.6		33.3		17.5		0	1
Distributed Angle Changes	0		−18.1		−15.7		−10.9		−6.7		−2.9			$1/(E\ I_o)$
Assumed Average Slope, Part (1)		38.1		20.0		4.3		−6.6		−13.3		−16.2		$\lambda/(E\ I_o)$
Trial Deflection, Part (1)	0		38.1		58.1		62.4		55.8		42.5		26.3	$\lambda^2/(E\ I_o)$
Deflection, Part (2)	0		−1.5		−1.3		−0.9		−0.6		−0.2		0	$\lambda^2/(E\ I_o)$
Linear Correction to Deflection	0		−4.4		−8.8		−13.2		−17.5		−21.9		−26.3	$\lambda^2/(E\ I_o)$
Resultant Deflection, w'_a	0		32.2		48.0		48.3		37.7		20.4		0	$\lambda^2/(E\ I_o)$
Final Deflections w_a Giving Same Deflections w'_a	0		32.4		48.4		48.9		38.1		20.6		0	$\lambda^2/(E\ I_o)$
Final w_o	0		143.0		189.8		179.7		135.2		71.7		0	$\lambda^2/(E\ I_o)$
Final Moments	180		185.7		167.4		134.9		93.8		47.9		0	1
Final End Slopes	212.9												−73.0	$\lambda/(E\ I_o)$

(*b*) Deflections and End Slopes for $M_B = 180$, $P = 0.25\ \frac{E\ I_o}{\lambda^2}$

	1	2	3	4	5	6	7	Common Factors
Final Deflections w_a Giving Same Deflections w'_a	0	19.1	30.4	32.4	26.4	14.7	0	$\lambda^2/(E\ I_o)$
Final w_o	0	67.6	113.1	129.4	115.4	71.7	0	$\lambda^2/(E\ I_o)$
Final Moments	0	46.9	88.2	122.3	148.8	167.9	180	1
Final End Slopes	72.7						−84.9	$\lambda/(E\ I_o)$

FIG. 41.—DEFLECTIONS AND BENDING MOMENTS FOR END MOMENTS COMBINED WITH AXIAL COMPRESSION



(*a*) Stiffness and Carry-Over Factors for Left End

			Common Factors
212.9	End Slopes, $M_A = 180$	−73.0	$\lambda/(E\ I_o)$
−62.5	End Slopes, $M_B = -0.861 \times 180$	73.0	$\lambda/(E\ I_o)$
150.4	Total Slope, $M_A = 180$, $M_B = -155$	0	$\lambda/(E\ I_o)$

Carry-Over Factor $= -0.861 = C_A$

$$\text{Stiffness} = \frac{180}{150.4}\ \frac{E\ I_o}{\lambda} = 1.195\ \frac{E\ I_o}{\lambda} = K_A$$

(*b*) Stiffness and Carry-Over Factors for Right End

			Common Factors
72.7	End Slopes, $M_B = 180$	−84.9	$\lambda/(E\ I_o)$
−72.7	End Slopes, $M_A = -0.341 \times 180$	24.9	$\lambda/(E\ I_o)$
0	Total Slope, $M_B = 180$, $M_A = -61.5$	−60.0	$\lambda/(E\ I_o)$

Carry-Over Factor $= -0.341 = C_B$

$$\text{Stiffness} = \frac{180}{60.0}\ \frac{E\ I_o}{\lambda} = 3.00\ \frac{E\ I_o}{\lambda} = K_B$$

(*c*) Rigidity Factor

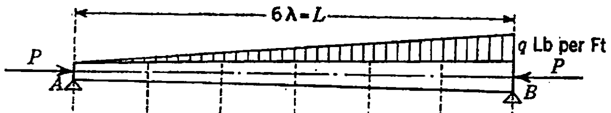
End Moments from Settlement, Δ, of End B, A and B Restrained from Rotation

$$-0.371\ \frac{E\ I_o\ \Delta}{\lambda^2} = K_A(1 + C_A)\ \frac{\Delta}{6\ \lambda^2} \qquad K_B(1 + C_B)\ \frac{\Delta}{6\ \lambda^2} = 0.671\ \frac{E\ I_o\ \Delta}{\lambda^2}$$

$$\text{Rigidity} = \text{Shear when } \Delta = 1 = \frac{(0.371 + 0.671)}{6\ \lambda}\ \frac{E\ I_o}{\lambda^2} = 0.1739\ \frac{E\ I_o}{\lambda^3}$$

FIG. 42.—DETERMINATION OF ELASTIC CONSTANTS

Determination of the elastic constants requires the computation of the end slopes at A and B (Fig. 41) due to end moments applied separately, first at A, then at B, including the effect of the given axial load, P, in both cases, assuming the beam to be simply supported. Initial deflections, w_i, and corresponding end slopes were obtained from Figs. 10(c) and 11(b). The first trial is shown in detail for end moment M_A, the intermediate trials being omitted. Final results only are shown for end moment M_B.



I/I_o

(a) Deflections for Triangular Load with $P = 0$

	1		2		3		4		5		6		7	Common Factors
Ordinates to Load Diagram	0		−30		−60		−90		−120		−150		−180	$q/180$
Assumed Average Shear, Part (1)		180		150		90		0		−120		−270		$q\lambda/180$
Trial Moments, Parts (1) and (2)	0		180		330		420		420		300		30	$q\lambda^2/180$
Linear Correction to Moments	0		−5		−10		−15		−20		−25		−30	$q\lambda^2/180$
Ordinates of Moment Diagram	0		175		320		405		400		275		0	$q\lambda^2/180$
Ordinates to Angle-Change Diagram	0		−87.5		−106.7		−101.3		−80.0		−45.8		0	$q\lambda^2/(180\,E\,I_o)$
Assumed Average Slope, Part (1)		237.5		150.0		43.3		−58.0		−138.0		−183.8		$q\lambda^3/(180\,E\,I_o)$
Trial Deflections, Part (1)	0		237.5		387.5		430.8		372.8		234.8		51.0	$q\lambda^4/(180\,E\,I_o)$
Deflections, Part (2)	0		−7.3		−8.9		−8.4		−6.7		−3.8		0	$q\lambda^4/(180\,E\,I_o)$
Linear Correction to Deflections	0		−8.5		−17.0		−25.5		−34.0		−42.5		−51.0	$q\lambda^4/(180\,E\,I_o)$
Deflections, w_i	0		221.7		361.6		396.9		332.1		188.5		0	$q\lambda^4/(180\,E\,I_o)$
End Slopes	239.1 = 221.7 + 17.4												−188.5 − 8.1 = −196.6	$q\lambda^3/(180\,E\,I_o)$

(b) Deflections for Triangular Load with $P = 0.25 \frac{E\,I_o}{\lambda^2}$

	1		2		3		4		5		6		7	Common Factors
Deflections, w_i	0		110.9		180.8		198.5		166.1		94.3		0	$q\lambda^4/(90\,E\,I_o)$
Assumed w_a	0		38.1		62.1		68.4		57.2		32.4		0	$q\lambda^4/(90\,E\,I_o)$
$w_o = w_i + w_a$	0		149.0		242.9		266.9		223.3		126.7		0	$q\lambda^4/(90\,E\,I_o)$
Moments due to P	0		37.3		60.8		66.9		55.9		31.7			$q\lambda^2/90$
Distributed Angle Changes	0		−18.7		−20.3		−16.7		−11.2		−5.3		0	$q\lambda^2/(90\,E\,I_o)$
Assumed Average Slope, Part (1)		38.7		20.0		−0.3		−17.0		−28.2		−33.5		$q\lambda^3/(90\,E\,I_o)$
Trial Deflections, Part (1)	0		38.7		58.7		58.4		41.4		13.2		−20.3	$q\lambda^4/(90\,E\,I_o)$
Deflections, Part (2)	0		−1.6		−1.7		−1.4		−0.9		−0.4		0	$q\lambda^4/(90\,E\,I_o)$
Linear Correction to Deflections	0		3.4		6.8		10.2		13.5		16.9		20.3	$q\lambda^4/(90\,E\,I_o)$
Deflections, w_a	0		40.5		63.8		67.2		54.0		29.7		0	$q\lambda^4/(90\,E\,I_o)$
Final Deflections w_a Giving Same Deflections w'_a	0		40.7		63.8		67.0		53.8		29.5		0	$q\lambda^4/(90\,E\,I_o)$
Final w_o	0		151.6		244.6		265.5		219.9		123.8		0	$q\lambda^4/(90\,E\,I_o)$
Final Moments	0		125.4		221.3		269.0		255.0		168.4		0	$q\lambda^4/90$
Final End Slopes	164.2												−128.6	$q\lambda^3/(90\,E\,I)$

FIG. 43.—DEFLECTIONS AND BENDING MOMENTS FOR TRIANGULAR LOAD COMBINED WITH AXIAL COMPRESSION

Fig. 42 illustrates the determination of stiffness and carry-over factors and rigidity from the results of Fig. 41 by a method similar to that of Fig. 11(c). (The formula for end moments in Fig. 42(c) is easily derived.[34])

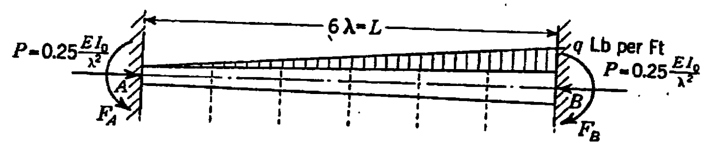
For determining fixed-end moments the additional quantities needed are the end slopes of the beam, assumed simply supported and subjected to the given lateral and axial load. Fig. 43 shows the calculation of these values for a triangular lateral loading condition, intermediate trials being omitted.

The computation of fixed-end moments, shown in Fig. 44(a), utilizes the data of Figs. 42(a), 42(b), and 43 and the same general principles previously

[34] "One Story Frames Analyzed by Moment Distribution," *Concrete Information Bulletin No. S T 48*, Portland Cement Assn., April, 1941, pp. 8 and 9.

applied in the determination of the elastic constants. In Fig. 44(b) the resultant deflections and bending moments for fixed ends and axial load are tabulated.

As a check, the bending moments of Fig. 44(b) are used to obtain the results of Fig. 44(c), the computed deflections and end slopes differing from those of Fig. 44(b) by a maximum of about 2%. Much of the work was done with a 5-in. slide rule, the remainder with a 10-in. slide rule.



(a) Fixed-End Moments

At A		At B	Common Factors
1.824	End Slopes, Beam Simply Supported (Fig. 43(b))		
− 1.824	End Rotation Producing Fixity at A, with B Clamped	− 1.427	$q\lambda^3/(E\,I_o)$
	Resulting End Moments	0	$q\lambda^3/(E\,I_o)$
$-2.180\,q\lambda^2 = 1.195 \frac{E\,I_o}{\lambda}\left(-1.824 \frac{q\lambda^3}{E\,I_o}\right)$		$-2.180\,q\lambda^2(-0.861) = 1.879\,q\lambda^2$	
0	End Rotation Producing Fixity at B, with A Clamped	1.427	$q\lambda^3/(E\,I_o)$
	Resulting End Moments		
$1.460\,q\lambda^2 = -4.281\,q\lambda^2(-0.341)$		$3.00 \frac{E\,I_o}{\lambda}\left(1.427 \frac{\lambda^3}{E\,I_o}\right) = -4.281\,q\lambda^2$	
	Fixed-End Moments		
$-0.720\,q\lambda^2 = F_A$		$F_B = -2.402\,q\lambda^2$	

(b) Resultant Deflections and Bending Moments
$P = 0.25\,E\,I_o/\lambda^2$ in All Cases

	1	2	3	4	5	6	7	Common Factors
Deflections due to F_A	0	−103	−137	−130	−98	−52	0	$q\lambda^4/(180\,E\,I_o)$
Deflections due to F_B	0	−163	−272	−311	−278	−173	0	$q\lambda^4/(180\,E\,I_o)$
Deflections due to $F_A + F_B$	0	−266	−409	−441	−376	−225	0	$q\lambda^4/(180\,E\,I_o)$
Deflections due to Triangular Load	0	303	489	531	440	248	0	$q\lambda^4/(180\,E\,I_o)$
Resultant Deflections	0	37	80	90	64	23	0	$q\lambda^4/(180\,E\,I_o)$
Resultant End Slopes	0						0	$q\lambda^3/(180\,E\,I_o)$
Moments due to F_A	−130	−134	−121	−98	−68	−35	0	$q\lambda^2/180$
Moments due to F_B	0	−113	−212	−294	−358	−404	−433	$q\lambda^2/180$
Moments due to $F_A + F_B$	0	−248	−333	−392	−426	−440	−433	$q\lambda^2/180$
Moments due to Triangular Load	0	250	443	538	510	337	0	$q\lambda^2/180$
Resultant Moments	−130	2	110	146	84	−103	−433	$q\lambda^2/180$

(c) Check Values of Deflections and End Slopes Obtained from Above Moments

	1	2	3	4	5	6	7	Common Factors
Deflections	0	36.2	79.4	88.8	63.5	22.6	0	$q\lambda^4/(180\,E\,I_o)$
End Slopes	−3.0						−0.9	$q\lambda^3/(180\,E\,I_o)$

FIG. 44.—FIXED-END MOMENTS, DEFLECTIONS, AND BENDING MOMENTS FOR TRIANGULAR LOAD COMBINED WITH AXIAL COMPRESSION

In all cases where the value of w_a for the first trial was computed from Eq. 4, a total of two trials gave a value of w'_a so close to the value of w'_a from a third trial that the extra trial was unnecessary. The effect of omitting deflections w_a, computed from Eq. 4, was to increase the required number of trials to three.

In comparison with other procedures devised for this type of problem, Professor Newmark's method affects savings in time and labor, gives equally good results, and is more easily checked.

I. OESTERBLOM,[35] M. AM. SOC. C. E.—The "numerical procedure," presented by Professor Newmark, for computing a variety of important elements in structural design constitutes quite an important tool in the workshop of the practicing engineer. Fundamentally there is nothing new in his basic idea, but it has taken both vision and a live imagination to see how neglected this idea was and how extensively it might be put to work.

As very often is the case, the father of a new method knows his child so well that he can describe it only indifferently. Thus it would have added to the usefulness of the paper, if the relationship between the differential equations referred to in the Synopsis and the new method had been more clearly outlined or described. This would have given the reader a better chance to generalize and extend the procedure. It also would have helped to make reading and understanding easier if there had been fewer: "* * * the calculations are self-explanatory." The procedure can be discovered after some work; but if new ideas are to be spread and not forgotten they must first be promoted. To use an old sales slogan: "A new thing will not sell itself; it requires a good sales talk." At least one illustration, perhaps two or three, should have been explained, point for point, with nothing omitted. Then the remainder would have followed easily, and new problems also could have been set up by the novice.

The ultimate service and grace of the method are the same either way, but enthusiasm would have been greater if the invitation to a delightful program would have been more convincing. A delightful program it seems to be, if one may judge from the many fields to which the method has been extended—moments, slopes, deflections, point loads, uniform loads, variable loads, axial loads, critical loads, and buckling loads. What more can mere man desire—in one single paper!

The reaction formulas for the variable loadings by Nádai and Southwell are good to have; and better yet it is good to be shown how they may be used to advantage by the Newmark method. The buckling formulas and how they are to be used are equally interesting.

The writer has not yet had a chance to apply Professor Newmark's method to any commercial problems, but he can well remember many problems from his past experience for which he would have been grateful to have this new information; and he feels certain that many of the younger engineers will be equally grateful when they are faced with similar problems.

C. W. DUNHAM,[36] M. AM. SOC. C. E.—The method of solving certain problems of deflections and buckling loads by successive approximations, presented in this paper, is based on the obvious and sensible idea that moments and loads cause deflections and deformations of the members subjected to them, and, as so often stated by Hardy Cross, M. Am. Soc. C. E., and others, both statics and geometry must be satisfied. In other words, the deformations and the causes of those deformations must be consistent with each other. If a shape of the deflected member is assumed, but moments accompanying those deflections are not the same as the moments acting on the member, then of course the assumed shape is wrong. However, it is obvious that sufficient data may be found from the first trial to make a better guess next time, thus arriving at a shape that is more nearly correct.

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As for the practicability and importance of the author's method, the writer has tried to look upon it as a tool for the designer to use and has tried to see it through the eyes of the ordinary man in a typical engineering office.

If it is desired to introduce a new method of analysis, or a modification of an established one, many things must be considered, some of which are:

1. If the ordinary reader cannot readily grasp the general features and if he thinks that the procedure is very complicated, he is not likely to give it serious study.

2. If he cannot check parts of the calculations when he does study them seriously, he will not be sure that the method can be trusted in new creative work of his own; hence he will not use it.

3. If he is to learn to use a method as a tool in his work, it must be simple, easy to understand, general in its application, and easy to remember and to apply.

4. If he is to develop facility in handling the procedure, he must apply it. This he will seldom do unless the method, as a tool, is applicable for solving problems with which he frequently comes in contact and which he must solve.

The writer will try to state his reactions to Professor Newmark's proposed method by discussing how it seems to meet the four preceding requirements:

1. A study of the paper gives the impression that the presentation makes the method appear more complicated than it really is. It is natural for any one who writes to believe that what is obvious to him is also obvious to the reader. However, such is not the case in many instances. It is unfortunate that the author did not give a little more explanation of the tie-up between his use of the "fictitious stringers" in computing moments in beams and the application of this procedure to the calculations of slopes and deflections when angle changes are used as loads. Of course, the explanation given by Professor Cross and N. D. Morgan, M. Am. Soc. C. E., is cited, but one illustrative problem showing this basic method more in detail would help to make the paper more complete in itself although risking repetition to the expert theorist. By its very nature, the paper is one that cannot be read casually and yet be appreciated. If the difficulties which it avoids were called to the reader's attention more forcefully, it might encourage him to give the paper the serious study which it deserves.

2. The writer decided to check most of the detailed work in the illustrative problems. In doing so, he found many "hops, skips, and jumps" which are likely to be confusing to a student—or, at least, hurdles which may handicap him in developing complete understanding of the work and confidence in his own ability to apply it independently. Some points which may be useful to others are listed below:

(a) In Fig. 6, the symmetry of shape and the fact that the tangent to the neutral curve is horizontal at the center of the span enables one to split the concentrated angle change of − 46 into two equal parts, the left one being positive because the tangent in that section slopes downward toward the right whereas the other side is the opposite. In Fig. 6, also, it might be advisable, as a first case, to label the last two lines of the calculations for deflection as follows: Average Slope (Shear); Deflection (Moment).

(b) A more detailed solution would be helpful in Fig. 7.

(c) In Fig. 10(a), the value 69.6 in the computation of the total end slope is derived by using Fig. 5(a) with values of the ordinates to the angle-change diagram.

(d) In Fig. 18(a), it might be well to give the reader some idea of how to make his first guess of the assumed average slope, Part 1. Here a good assumption would be about half of the sum of the distributed angle changes. However, in Fig. 19(a), trial appears to be the only way to determine a suitable starting value.

(e) The force 2 P in Fig. 19(a) should be shown clearly to be applied at the fourth division point. It would also be helpful to show the following for the computation of moments, as for the value 175 at the first division point: The rotational moment due to the deflection of the point of application of 2 P is 2 P (1,000 a) clockwise; the end reaction (vertical) due to this moment is $2\,P\,(1{,}000\,a) \div 10\,\lambda = 200\,\frac{P\,a}{\lambda}$ acting counter-clockwise; and the moment at the first division

$$= P\,(375\,a) - \left(200\,\frac{P\,a}{\lambda}\right)\lambda = 175\,P\,a.$$

(f) It would be helpful if the author used specific numerical cases for Figs. 19 and 20, showing how to derive a scale for the value of a.

3. The general method shown in the paper is really simple and readily understandable. However, it seems desirable to adopt one standard set of details of procedure rather than to show special short-cut methods for particular cases. A thorough understanding of the basic method is desirable. If it can be mastered thoroughly, one need seldom worry about the fact that a modification of it may be more efficient in a special case. It is wise to have the men in an office able to use basic procedures correctly and to be able to check each others' work without undue disputes as to refinement of methods. Furthermore, the assumptions that must be made in practical work regarding magnitudes of loads, their directions, their points of application, conditions of end restraint, original straightness of members, span lengths, and the properties of the materials introduce so many approximations that it seems unnecessary in most cases to refine the calculations for curvature of loading diagrams when the straight-line approximations may be far more correct than the basic data from which the computations are started. Would it be more beneficial to confine the method to the use of substitute straight-line moment diagrams and to illustrate its application more fully by numerical cases? This might make it very easy to remember and to apply. The arrangement of the calculations (bookkeeping) is very simple and satisfactory.

4. It appears that the proposed method is useful in determining the deflected shapes of members under certain assumptions. However, this is not generally important except in special cases. As for its use in designing columns, it seems to the writer that the buckling loads are seldom of great interest to the designer of ordinary structures although they may be important in the design of machinery, airplanes, and similar structures requiring special refinement and care in their design. In ordinary structural work, the designer generally selects a tentative member because of various practical reasons which make it seem desirable, or he chooses one by using various approximations, and then he analyzes it to prove that it is satisfactory for the loading conditions under which it must act. Generally, he is not interested in its ultimate buckling load. Therefore, the field of usefulness of the proposed method may be rather limited.

However, the author is to be congratulated upon developing such a simple approximation for the solution of problems. When they do arise and the designer must meet them, he will need such a handy tool and he will need it badly.

N. M. Newmark,[37] Assoc. M. Am. Soc. C. E.—The suggestions, criticisms, and examples of the application of the numerical procedure to specific problems that have been given by the discussers of the paper are appreciated.

The use of the procedure to obtain elastic constants for beams is illustrated by Mr. Johnston. His suggested procedure is applicable to problems in which deflections are not desired, but where end slopes are required and are determined from the equivalent concentrated angle-change loads. In this way the procedure can be used to determine relatively accurate values for elastic constants with a comparatively small number of segments in the length of the beam.

Influence lines for fixed-end moment may be obtained from the deflection curve for a beam with a unit rotation at the end, by use of the so-called Müller-Breslau principle. Consequently, it may be convenient often to compute deflections even in problems such as those considered by Mr. Johnston.

Professor Ketchum has suggested a way of estimating the additional deflection due to the end thrust when a bar is subjected to lateral load and end thrust. His procedure amounts to assuming that the buckling configuration of the bar is the same in shape as the deflection curve due to the lateral load. For more or less uniform distributions of loading his approximation is reasonably good and leads to fairly accurate results.

In order to correlate Professor Ketchum's procedure with that suggested by the writer, one can determine an approximate value for the critical load from Eq. 19b, as the average ratio between the moment due to the lateral load and the deflection due to the lateral load. In order to estimate the additional deflection to be used in the numerical procedure, Eq. 4 can be used with the approximate value of the critical load determined from Eq. 19b. In many cases this will permit a problem to be solved without going through the routine of determining the critical load first, and therefore the suggestion is an important addition to the paper.

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The use of simultaneous equations in solving the numerical problem was mentioned by Professor Wilbur. Although simultaneous equations can be written for the calculation of the deflections in a bar loaded with lateral loads and end thrusts, in general it is neither convenient nor desirable to write these equations. In the few cases where the numerical procedure does not converge, it may be desirable to have recourse to the equations. The following procedure will illustrate the way in which the equations may be written.

Let $m - 1$, m, and $m + 1$ be three neighboring points on the bar, at a distance λ apart. The following notation is used, in which the subscripts refer to the particular point on the bar:

$(w_i)_m$ = deflection due to the lateral load and the initial configuration, at point m;
$(w_a)_m$ = additional deflection at point m due to the axial loads; and
$(E\,I)_m$ = value of the product $E\,I$ at the point m.

The change in slope at point m is equal to the equivalent concentrated angle change at m; but the added change in slope is the quantity $\frac{(w_a)_{m+1} - 2(w_a)_m + (w_a)_{m-1}}{\lambda}$. If the moment at point m due to the axial load P is $P[(w_i)_m + (w_a)_m]$, and if the angle-change curve is a smooth curve, the equivalent concentrated angle change at point m is the quantity

$$-\frac{P\,\lambda}{12}\left[\frac{(w_a)_{m+1} + (w_i)_{m+1}}{(E\,I)_{m+1}} + 10\,\frac{(w_a)_m + (w_i)_m}{(E\,I)_m} + \frac{(w_a)_{m-1} + (w_i)_{m-1}}{(E\,I)_{m-1}}\right]$$

For axial loads, applied at other points than at the end of the bar, the expression for moment is changed, and, where the angle-change curve is not a smooth curve, the equivalent concentrated angle changes are somewhat different in form. The procedure for such cases is not essentially different from that described herein.

Equating the change in slope and the equivalent concentrated angle change at each point on the bar leads to a set of linear equations for the unknowns w_a for each point, since all the other quantities are known.

The equations are of the following form:

$$-\left[1 + \frac{P\,\lambda^2}{12(E\,I)_{m+1}}\right](w_a)_{m+1} + \left[2 - \frac{10\,P\,\lambda^2}{12(E\,I)_m}\right](w_a)_m - \left[1 + \frac{P\,\lambda^2}{12(E\,I)_{m-1}}\right]$$
$$\times\,(w_a)_{m-1} = \frac{P\,\lambda^2}{12(E\,I)_{m+1}}(w_i)_{m+1} + \frac{10\,P\,\lambda^2}{12(E\,I)_m}(w_i)_m + \frac{P\,\lambda^2}{12(E\,I)_{m-1}}(w_i)_{m-1}\,.\,.\,(41)$$

Since there are as many equations as unknowns, Eqs. 41 can be solved for the additional deflections w_a.

When the values of w_i are zero at all points, there is a problem of pure buckling, and the equations are homogeneous equations which involve only the unknowns w_a and P, with the constant terms being zero. In order that the set of equations may have a solution different from the obvious one with all the quantities w_a being zero, the determinant of the coefficients must vanish. This leads to an algebraic equation for P in which P appears to some power equal to the number of points on the bar that can deflect. The solution of the

equation yields the critical values of P corresponding to the different modes of buckling. The lowest value of P is generally the only one of interest.

From this discussion the reader may be able to see why it is desirable to use the method of successive approximations which avoids dealing with the equations.

Professor Wilbur has given formulas for the equivalent concentrated load when the loading curve is divided into segments unequal in length. These formulas are expressed by Eqs. 23*a* and 23*b* and refer to Fig. 25. In some applications it is desirable to have a formula for the magnitude of R_{ab}, which was not given by Professor Wilbur. This quantity is defined by the following equation, using the same notation that was used by Professor Wilbur.

$$R_{ab} = \frac{l_1}{12\,l_2(l_1 + l_2)}\left[a\,l_2(3\,l_1 + 4\,l_2) + b(l_1 + l_2)(l_1 + 2\,l_2) - c\,l^2_1\right]\,.\,.\,(42a)$$

The equivalent concentration at point b, namely, R_b, which is equal to $R_{bc} + R_{ba}$ may be simplified to the following form:

$$R_b = \frac{l_1 + l_2}{12}(a + 4\,b + c) + \frac{l^2_2}{12\,l_1}(b - a) + \frac{l^2_1}{12\,l_2}(b - c)\,.\,.\,.\,.\,(42b)$$

In general, it is not convenient to use different lengths of segments in the same problem although in some cases it may be desirable to use segments of one particular length for part of the beam and of another length for the remainder. In such cases, one can use the formulas that apply to segments of constant length in order to obtain the equivalent concentrations, by working from both sides at the point where the segments change in length.

The use of segments of different length always can be avoided by the simple expedient of calculating the proper equivalent concentrations at the chosen division points to account for the actual conditions in the segments between division points. A simple example of such a procedure is shown in Fig. 45, in which moments are calculated for a given distribution of loading. The same type of procedure is used for calculating deflections from angle changes.

Regarding the calculation of the critical load for a bar composed of segments of different moments of inertia, Professor Wilbur has given a correct analysis by means of the usual formal solution of the differential equation. However, it is precisely such an analysis that the writer sought to avoid with the numerical procedure. In a relatively simple problem it is not too complicated a matter to solve the differential equation. For complicated variations in the moment of inertia, however, the formal solution of the differential equation may not be convenient. Even in this problem, the determination of the critical load requires the solution of a transcendental equation which may take considerable time for an engineer unaccustomed to solving such problems. For practical purposes the results so obtained are not of any greater accuracy than those given by the numerical procedure.

To summarize: Special problems can be solved by use of procedures such as those illustrated by Professor Wilbur, but a different technique is required in each case. With the numerical procedure only one technique is required and it is applicable to all problems.

Mr. Stewart illustrates the use of the numerical procedure in computing constants to be used in his particular method of frame analysis. The writer cannot concur with his statement concerning "basic" constants of beam flexure. The distinction between "basic" and "derived" constants depends entirely upon the point of view of the person making the analysis. In one pro-

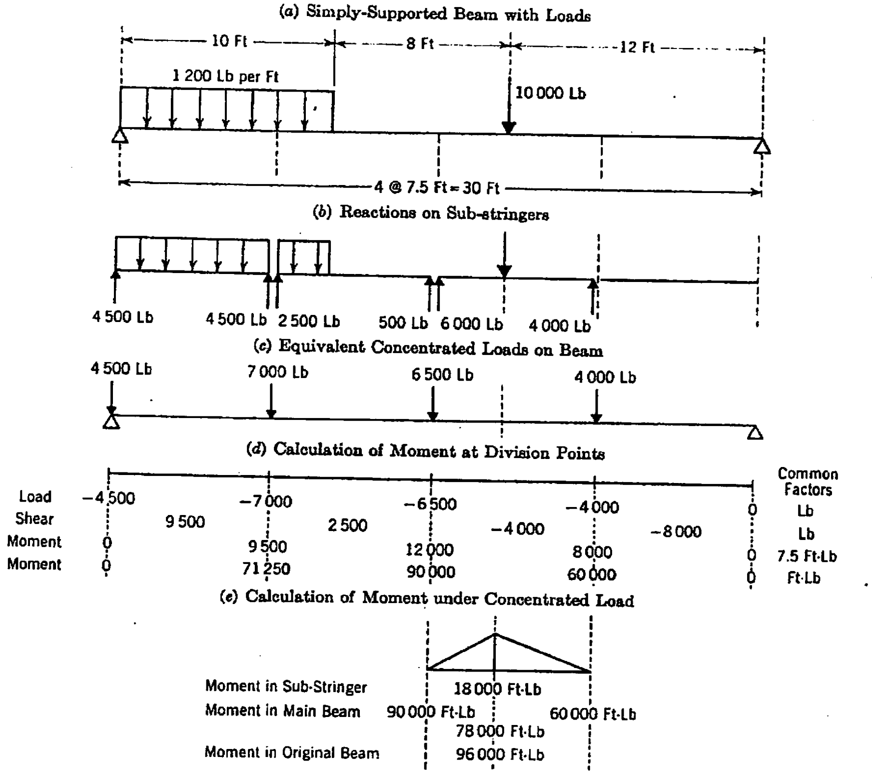

Fig. 45.—Treatment of Loadings with Discontinuities Between Division Points

cedure certain constants may be "basic" and in another procedure entirely different constants may be "basic." From the writer's point of view, the stiffness and carry-over factors in Professor Cross' moment distribution procedure are fundamental and basic constants from which any other constants used in other methods of analysis can be derived by simple equations. One can determine these constants by experiment or by analysis—by experiment, preferably, in cases where there are marked departures from Hooke's law or where there is slip at joints or other phenomena not readily amenable to analytical treatment.

However, it was not the object of the paper to discuss the merits of different methods of analysis of frames. The procedure can be used, as Mr. Stewart himself has shown, to compute the beam constants for any of the methods of analysis.

Mr. Fraenkel's use of the numerical procedure in his analysis of a derrick boom is interesting. It can be seen from his analysis that the effect of the longitudinal force in the boom on its deflection is relatively negligible. This might be estimated in advance, as a very rough preliminary estimate of the critical load for the boom could have been made by assuming some equivalent constant moment of inertia, say, that in the central section. Since the ratio of the actual end thrust to the critical end thrust is very small, the effect of the end thrust on the deflection must be small also.

In connection with Mr. Fraenkel's discussion, it should be noted that the effect of longitudinal thrust cannot be superposed on the effect of lateral loads. Moments must be computed for each complete loading. That is, the effect of live loading cannot be determined from the live loading alone, but must be determined by computing the difference between the effect of the combined dead and live load and the effect of the dead load alone.

The writer wishes to thank Professor Niles for his kind remarks concerning the numerical procedure. As Professor Niles states, for continuous beams the numerical procedure is not directly applicable without some additional work. However, one can compute constants for use in any of the standard methods of analysis for continuous beams—for example, moment distribution, the three moment equation, or the slope deflection method. Then, by use of any of these methods, one can compute the bending moments over the supports. When the end moments are known, the deflection curve for each span can be obtained if it is desirable to do so. When the members are subjected to axial thrusts as well as lateral loads, the calculation of the beam constants is facilitated greatly by the use of the numerical procedure. Mr. Williamson has indicated ably an application to such a problem.

It is always possible to use segments of constant length in the analysis, even when unequally spaced concentrated loads are present. This is illustrated by the problem solved in Fig. 45. Where discontinuities in section occur between division points, one usually can estimate reasonably well the equivalent concentration to use at the division point. When formulas are desired, they can be derived readily by consideration of the sub-stringer between division points. If one finds the equivalent concentration on the sub-stringer at the point of discontinuity (usually by means of the equations applying to segments of unequal length), one then can determine the sub-stringer reactions due to the equivalent interior concentration and proceed from that point with segments of equal length. However, in most cases, taking short segments may make it unnecessary to consider the discontinuities between the division points.

In most practical problems one can use the short-cut procedure for making the correction to the moment diagram without serious error. It is only when a relatively small number of divisions are used that it is necessary to compute the equivalent concentrations and to use them instead of the ordinates to the loading or angle-change curve.

As Professor Niles has indicated, the formulas for equivalent concentrations require division by certain factors. In his numerical problems the writer has taken this factor as a part of the "common factor." In many cases it is very convenient to do so; in others it is worth while to compute the actual equivalent concentration merely to prove to one's self that this quantity does not differ materially from the ordinate to the actual distribution curve.

The writer has no liking for any particular convention of signs. In his own work he uses different conventions at different times. Since deflections are considered positive downward in most engineering literature, they were considered positive downward in the paper. It is simple enough to change the sign convention to one that seems better for a specific purpose if one wishes to do so.

Professor Niles objects to the use of the term "angle change." However, it seems to be descriptive enough of what is meant and it avoids somewhat cacophonous terminology, as for example, "curvature" curve. The term "angle change" is used among structural engineers, and the writer can claim no credit nor take any blame for developing the name. It would be just as acceptable to call the quantity the "rate of slope change."

The numerical value that Professor Niles refers to in Fig. 15(*a*) was computed by means of the following formula: $-404.90 = -\frac{1}{24}(7 \times 803.6 + 6 \times 513.5 - 1 \times 0 + 7 \times 80.36 + 6 \times 91.10 - 1 \times 97.75)$. The writer used this formula rather than the one that Professor Niles used since there is a discontinuity in section of the beam at this particular point. It will be noted that Professor Niles' formula requires multiplying or dividing certain ordinates to the angle-change curve by 10, whereas the foregoing expression uses only the actual ordinates without modification. As Professor Niles has indicated, the difference is of no practical consequence.

The use of the procedure to compute influence lines is illustrated ably by Mr. Weiss. Since deflections of a structure subjected to certain distortions are the influence values desired, the procedure leads directly to influence ordinates. In this connection the influence for moment at a fixed end of a beam can be obtained directly from the calculations of the elastic constants. For example, in Fig. 10 the influence for moment at the left end of the beam, when the left end is fixed and the right end is simply supported, is obtained by dividing the deflections of the various points by the slope at the end of the beam. That is, the influence ordinate for moment at the left end due to a unit load at the center is $\frac{131.3\,\lambda}{181.0} = 0.725\,\lambda$.

When computing influence lines for continuous beams, the writer generally prefers the procedure of introducing the required discontinuity in the span considered, finding the fixed-end moments, and distributing these moments to obtain the moments over each support. Then the deflections of the structure in each span can be determined readily. In certain instances, however, it might be more convenient to use the procedure given by Mr. Weiss, although the combination of the deflections due to the reactions at the various points may lead to results which are the differences between large quantities and which, therefore, may be relatively inaccurate unless intermediate calculations are carried to a large number of significant figures.

Mr. Eremin has illustrated the use of the numerical procedure for a problem in which a bending moment is applied at an intermediate point in a beam.

With a beam of constant moment of inertia, such as Mr. Eremin has considered, the numerical values should be exact since the moment diagram is made up of straight-line parts. It should be exact even if the moment diagram were made up of parabolic segments.

Mr. Gossard uses the procedure to obtain constants for haunched beams with various types of haunches at the ends. His application of the procedure is interesting. However, in view of the complications involved in such a procedure, it seems preferable to the writer to compute the constants for the beam by the numerical procedure directly, rather than to fit the different parts together. Whatever is done will depend upon the designer's personal preference. Mr. Gossard undoubtedly will find extensive use for the tables and curves he has computed.

Mr. Williamson's able discussion is appreciated. His illustration of the calculation of elastic constants for a member carrying axial thrust is a valuable addition to the paper and suggests further applications in the field of aircraft stress analysis. It is not difficult to use the procedure to make analyses of beams and columns in which the stresses go beyond the elastic limit. One determines a relationship between moment and "angle change" for any specified thrust and any given moment, either by means of trial or by a systematic set of calculations which lead to graphical relationships in the form of curves. With these data known, the analysis proceeds in the customary manner, using the relationship between angle change and moment determined from the magnitude of the moment.

With regard to Mr. Oesterblom's comments, the writer would like to state that using the procedure—that is, the actual numerical computation of a problem—is the only way in which the procedure can be learned. That it can be learned in such a way the writer has verified in his teaching.

Professor Dunham has made a careful study of the paper and concludes, from the viewpoint of the ordinary man in a typical engineering office, that the field of usefulness of the method is limited. He would prefer, apparently, to have one basic procedure, without even minor modifications, to fit every case that might arise, although he anticipates use of the method only for structures requiring special refinement and care in their design.

The writer is glad to have the comments of one familiar with the problems of the designer. However, he feels that the average designer does not need all his work laid out for him in such a way that all he has to do is to fill out a form. In any case, the paper is written also for the man who decides what forms are to be filled out. There are many considerations of immense practical importance, besides the technique of analysis, involved in any design problem that would require calculations of the type contemplated in the paper. The average designer, capable of taking these things into account, is also capable of deciding whether to consider straight-line or curved loading diagrams or whether to use merely the ordinates at the division points without referring to any equivalent concentration at all. He has the formulas for any of these possibilities available to him. Exactly what he should do is merely a matter of technique. Professor Dunham may decide on a special technique for the problems that he or his organization ordinarily encounters. Neither the writer nor Professor Dunham can decide on a scheme that also would be best in an application to a problem arising in the design of an airplane, for example.

Regarding specific comments of Professor Dunham's, the writer would like to suggest the following:

It is certainly not obvious that one may find from his first trial deflection curve sufficient data to make a better guess next time. It is extremely fortunate, for most practical cases, that the derived deflection curve is a better approximation to the true curve than the first guess. However, such is not always the case, as is demonstrated by the problems shown in Figs. 19 and 20. Two entirely different possibilities need to be considered if a procedure of successive approximations does not converge to the correct result. First, the procedure may not converge at all, which will be an obvious warning to the designer that he must find an alternative method for his study of the problem; but the other possibility is that the procedure converges to the wrong answer, and the designer has no obvious way of knowing when this is the case. It is the writer's earnest hope that the types of problem in which such eventualities occur are illustrated sufficiently in the text.

It is unfortunate that the writer's presentation makes the method "appear more complicated than it really is." Professor Dunham's comments may help to alleviate matters somewhat. In considering the presentation of the material, the writer attempted to take into account the fact that similar procedures have been presented before and have been used by many people in numerous problems. Furthermore, the simple basis for the procedure appeared to be common knowledge. Only departures from more or less standard routines were described in detail.

Referring to Professor Dunham's comments under 2(*f*), the value of a is entirely immaterial. It can be taken as one unit, as in the other illustrative problems. It is taken as a general quantity in the figures referred to in order that the moments, which depend on a, should have the proper dimensions. It might have been more consistent to have used such a factor in all the problems where a deflection had to be assumed.

In closing, the writer again wishes to thank all of the discussers for their efforts to add to the value of the paper. His only contribution has been more or less a bookkeeping procedure for using a method of analysis that is relatively old and that has been revised successively by various writers up to the present time. There are countless variations of similar numerical procedures, many of which the writer studied in detail before he arrived at the conclusion that the particular procedure given in the paper led to greater accuracy with less numerical work than any of the others that came to his attention. Even when one does not use the part of the procedure that calls for a calculation of equivalent concentrated loads or angle changes, the results of the calculation are reliable if one takes a somewhat greater number of segments, say, ten to twelve, in the length of a beam. It is hoped that those who have occasion to deal with problems of the flexure of members subjected to direct stresses and lateral load will find the procedure useful and time saving.

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FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM

A SYMPOSIUM

With Discussion by Messrs. Edwin F. Wendt and L. T. Fleming, Anson Marston, William G. Atwood, George E. Goldthwaite, Nathan B. Jacobs, H. J. Flagg, Nelson Lee Smith, C. Beverley Benson, Paul T. Norton, Jr., H. L. Ripley, Carroll A. Farwell, Thomas R. Agg, Conde B. McCullough, W. V. Burnell, Roger L. Morrison, Wallace B. Carr, W. L. Waters, A. G. Mott, David A. Kosh, E. G. Walker, K. Lee Hyder, James T. Ryan, John C. Page, John S. Worley, John F. Miller, Fred Asa Barnes, Robert F. Legget and Erwin E. Hart, Kernan Robson, J. A. Walls, and Eugene L. Grant.

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